Consider a two-state quantum mechanical system with energies E_1 and E_2. Suppose it is acted upon by a time-independent perturbation represented by the matrix:

$$\delta V = a\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

where a is small compared to $E_1 - E_2$. If at time $t=0$, it is in state 1, find the probability that it is in state 2 after a time T.

we are not told the eigenstates of the unperturbed system. So, assume

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The perturbed energies are given by the eigenvalues of δV. So, we have

$$\det\begin{bmatrix} -\lambda & ia \\ -ia & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 = a^2 \Rightarrow \lambda = \pm a$$

Then the eigenstates are

$|\lambda = a\rangle$|

$$\begin{pmatrix} -a & ia \\ -ia & -a \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow -a\phi_1 + ia\phi_2 = 0 \Rightarrow \phi_1 = i\phi_2$$

$$\Rightarrow |\lambda = a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$|\lambda = -a\rangle$|

$$\begin{pmatrix} a & +ia \\ -ia & a \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow \phi_1 = -i\phi_2 \Rightarrow |\lambda = -a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Now, write $|1\rangle$ and $|2\rangle$ in terms of $|\lambda = \pm a\rangle$. That is,

$$|1\rangle = \frac{1}{\sqrt{2}}|\lambda = a\rangle + \frac{1}{\sqrt{2}}|\lambda = -a\rangle$$

$$|2\rangle = \frac{i}{\sqrt{2}}|\lambda = a\rangle - \frac{i}{\sqrt{2}}|\lambda = -a\rangle$$

Now apply the time evolution operator to the $t=0$ state, $|1\rangle$. That is,

$$|\psi(t=0)\rangle = |1\rangle$$

and

$$|\psi(t)\rangle = e^{-iHt}|1\rangle = e^{-iHt}\left[\frac{1}{\sqrt{2}}|\lambda=a\rangle + \frac{1}{\sqrt{2}}|\lambda=-a\rangle\right]$$

$$\Rightarrow \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-it(E_1+a)}|\lambda=a\rangle + \frac{1}{\sqrt{2}} e^{-it(E_1-a)}|\lambda=a\rangle}$$

The probability for it to be in state 2 after a time T is

$$|\langle 2|\psi(t)\rangle|^2 = \left|\left(\frac{-i}{\sqrt{2}}\langle\lambda=a| + \frac{i}{\sqrt{2}}\langle\lambda=-a|\right)\left(\frac{1}{\sqrt{2}} e^{-iT(E_1+a)}|\lambda=a\rangle + \frac{1}{\sqrt{2}} e^{-iT(E_1-a)}|\lambda=a\rangle\right)\right|^2$$

$$= \frac{1}{4}\left| +i\left(-e^{-iT(E_1+a)} + e^{-iT(E_1-a)}\right)\right|^2$$

$$= \frac{1}{4}\left| e^{-iTE_1}\left(-e^{-iTa} + e^{+iTa}\right)\right|^2$$

$$= \frac{1}{4}\left|2i\sin(Ta)\right|^2$$

$$\therefore \boxed{|\langle 2|\psi(t)\rangle|^2 = \sin^2(Ta)}$$