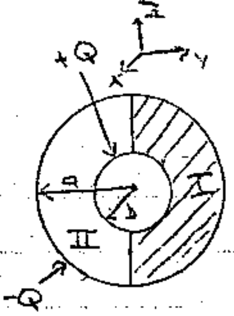
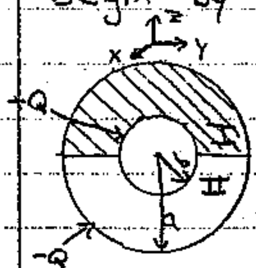


Fall 1998 #1

Two concentric spheres of inner radii a and b carry charges $\pm Q$. The empty space is half filled by a hemispherical shell of dielectric constant

a) Find the electric field everywhere between the spheres

→ Begin by considering the physically identical system shown below



The system has azimuthal symmetry, thus the potential has the general form.

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

However, the concentric spheres are conductors, thus the electric field must be normal to the surface. So for this case E_θ must equal zero. The $\hat{\theta}$ component of $\vec{E}$ is given by: $E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}$

Thus

$$E_\theta = \sum_{\ell=0}^{\infty}\left[-A_\ell r^{\ell-1} - B_\ell r^{-(\ell+1)-1}\right]\frac{\partial}{\partial\theta}P_\ell(\cos\theta)$$

yields

$$A_\ell r^{\ell-1} + B_\ell r^{-\ell-2} = 0 \qquad \Rightarrow \quad A_\ell = -B_\ell r^{-2\ell-1}$$

so separately at $r=a$ and $r=b$

$$A_\ell = -B_\ell (a)^{-2\ell-1} \qquad\qquad A_\ell = -B_\ell (b)^{-2\ell-1}$$

or

$$B_\ell (a)^{-2\ell-1} = B_\ell (b)^{-2\ell-1}$$

this condition is only satisfied by the trivial result $B_\ell = 0$. This means that for Φ to be nonzero, it cannot depend on Θ, and only $\ell = 0$ terms can appear

$$\Phi(r) = A + B r^{-1}$$

The electric field in regions I, and II is therefore

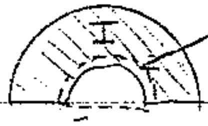
$$\vec{E}_1(r) = \frac{B_I}{r^2}\hat{r} \qquad\qquad \vec{E}_2(r) = \frac{B_{II}}{r^2}\hat{r}$$

where I, II refer to the region

To be more precise must answer (b) first.

b) Calculate the surface charge distribution on the inner sphere.

→ Apply Gauss's Law to include surface charge on inner sphere in par



for region I, Gauss's Law in dielectrics yields

$$D_I(\cancel{2}\pi r^2) = 4\pi(\cancel{2}\pi a^2 \sigma_I) \qquad \text{(the flat segment do contribute}$$

σ_I = surface charge on conductor

$$\vec{D}_I(r) = 4\pi\sigma_I \frac{a^2}{r^2}\hat{r} \quad \Rightarrow \vec{E}_I = D/\epsilon = \frac{4\pi\sigma_I}{\epsilon}\frac{a^2}{r^2}\hat{r}$$

the same argument in region II yields

$$E_{II} = 4\pi\sigma_{II}\frac{a^2}{r^2}\hat{r}$$

Now, we also know that at an interface, the tangential component of E must be continuous (the radial component in this case)

$$E_I = E_{II} \quad \Rightarrow \quad \sigma_I = \epsilon\sigma_{II}$$

we also know that $\sigma_I + \sigma_2 = Q$ so

$$2\pi a^2\sigma_{II} + 2\pi a^2\epsilon\sigma_{II} = Q$$

$$\boxed{\sigma_{II} = \frac{Q}{2\pi a^2(1+\epsilon)}} \qquad \boxed{\sigma_I = \frac{\epsilon Q}{2\pi a^2(1+\epsilon)}}$$

also the final answer for (a) is now.

$$\boxed{E_I = \frac{2Q}{(1+\epsilon)r^2}\hat{r}} \qquad \boxed{E_{II} = \frac{2Q}{(1+\epsilon)r^2}\hat{r}}$$

the field is the same in both regions

c) Calculate the polarization charge density induced on the surface of the dielectric at $r=a$.

$$\sigma_b = \vec{P}\cdot\hat{n} \qquad \text{where} \qquad P = \frac{D-E}{4\pi}$$

$$P = \frac{1}{4\pi}(\epsilon-1)\vec{E} = \frac{Q(\epsilon-1)}{2\pi(\epsilon+1)r^2}\hat{r}$$

so

$$\boxed{\sigma_b = \frac{Q(\epsilon-1)}{2\pi(\epsilon+1)a^2}}$$