Fall 1998 #10

Consider H atom with total orbital angular momentum squared $2\hbar^2$. A measurement of the z component of orbital angular momentum results in the value $+\hbar$. One then measures the orbital angular momen along the direction $\hat{n} = (\sin\Theta, 0, \cos\Theta)$. What are the possible results of the second measurement? What is probability as function of Θ that the result is $+\hbar$?

→ We are told that $J^2|\psi\rangle = 2\hbar^2|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle$

thus $l=1$, and we will be using the representation:

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_z = \hbar\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The given info states that z component of orbital angular momentum

Find the eigenvector for this state.

$$\hbar\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 0-1 & 0 \\ 0 & 0 & -1-1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 0a = 0 \\ -b = 0 \\ -2c = 0 \end{matrix} \Rightarrow |\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

the 2nd measurement occurs along the direction $(\sin\Theta, 0,$

so the representation matrix along this direction is given by:

$$D(J(\hat{n})) = \hat{n}\cdot D(J) \qquad \text{(Abers pg 96)}$$

$$D(J(n)) = \hbar\begin{pmatrix} \cos\Theta & \frac{1}{\sqrt{2}}\sin\Theta & 0 \\ \frac{1}{\sqrt{2}}\sin\Theta & 0 & \frac{1}{\sqrt{2}}\sin\Theta \\ 0 & \frac{1}{\sqrt{2}}\sin\Theta & -\cos\Theta \end{pmatrix}$$

find eigen values

$$\hbar\begin{vmatrix} \cos(\Theta)-\lambda & \frac{1}{\sqrt{2}}\sin\Theta & 0 \\ \frac{1}{\sqrt{2}}\sin\Theta & -\lambda & \frac{1}{\sqrt{2}}\sin\Theta \\ 0 & \frac{1}{\sqrt{2}}\sin\Theta & -\cos\Theta-\lambda \end{vmatrix} = 0$$

$$0 = (\cos\Theta - \lambda)\left(+\lambda\cos\Theta + \lambda^2 - \frac{1}{2}\sin^2\Theta\right) - \frac{1}{\sqrt{2}}\sin\Theta\left(\frac{-1}{\sqrt{2}}\sin\Theta\cos\Theta\right)$$

$$\left(\lambda\cos^2\Theta + \cancel{\lambda^2\cos\Theta} - \frac{1}{2}\sin^2\Theta\cos\Theta - \lambda^3 + \lambda\frac{1}{2}\sin^2\Theta\right) = 0$$

$$+\lambda\frac{1}{2}\sin^2\Theta$$

$$-\lambda^3 + \lambda(\sin^2\Theta + \cos^2\Theta) = 0$$

$$\lambda(1-\lambda^2) = 0 \qquad \Rightarrow \lambda = 0, \pm 1$$

possible results are $0, +\hbar, -\hbar$ for 2nd measurement

find eigenstate for $+\hbar$:

$$\hbar\begin{pmatrix} \cos\Theta - 1 & \frac{1}{\sqrt{2}}\sin\Theta & 0 \\ \frac{1}{\sqrt{2}}\sin\Theta & -1 & \frac{1}{\sqrt{2}}\sin\Theta \\ 0 & \frac{1}{\sqrt{2}}\sin\Theta & -\cos\Theta - 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} a(\cos\Theta - 1) + b\tfrac{1}{\sqrt{2}}\sin\Theta &= 0 \\ a\tfrac{1}{\sqrt{2}}\sin\Theta - b + c\tfrac{1}{\sqrt{2}}\sin\Theta &= 0 \\ b\tfrac{1}{\sqrt{2}}\sin\Theta - c(\cos\Theta + 1) &= 0 \end{aligned}\right\} \Rightarrow \quad b = a\frac{\sqrt{2}(\cos\Theta - 1)}{\sin\Theta}, \quad c = b\frac{\sin\Theta}{\sqrt{2}(\cos\Theta+1)}$$

$$|\phi\rangle = N\begin{pmatrix} 1 \\ \frac{\sqrt{2}(\cos\Theta - 1)}{\sin\Theta} \\ \frac{(\cos\Theta - 1)}{(\cos\Theta + 1)} \end{pmatrix}(\cos\Theta + 1) \Rightarrow N'\begin{pmatrix} \cos\Theta + 1 \\ \frac{\sqrt{2}(\cos^2\Theta - 1)}{\sin\Theta} \\ \cos\Theta - 1 \end{pmatrix} = \begin{pmatrix} \cos\Theta + 1 \\ -\sqrt{2}\sin\Theta \\ \cos\Theta - 1 \end{pmatrix}$$

$$N' = \frac{1}{\sqrt{\cos^2\Theta + \cancel{2\cos\Theta} + 1 + 2\sin^2\Theta + \cos^2\Theta - \cancel{2\cos\Theta} + 1}} = \frac{1}{2}$$

Thus,

$$|\phi\rangle_\hbar = \frac{1}{2}\begin{pmatrix} \cos\Theta + 1 \\ -\sqrt{2}\sin\Theta \\ \cos\Theta - 1 \end{pmatrix}$$

$$P_\hbar(\Theta) = |\langle\phi|\psi\rangle|^2 = \left|\frac{1}{2}(\cos\Theta + 1, -\sqrt{2}\sin\Theta, \cos\Theta - 1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right|^2$$

$$\boxed{P_\hbar(\Theta) = \left[\frac{1}{2}(\cos\Theta + 1)\right]^2 = \frac{1}{4}(\cos\Theta + 1)^2}$$