$F'98$ #10 Q.M.

Consider a hydrogen atom with total orbital angular momentum squared $2\hbar^2$. A measurement of the z-component of orbital angular momentum results in the value $+\hbar$. One then measures the component of orbital angular momentum along the direction $\hat{n}$, where $\hat{n} = (\sin\theta, 0, \cos\theta)$. What are the possible results of the second measurement? What is the probability (as a function of θ) that the result is $+\hbar$?

For an angular momentum, for $j=1$, in the usual representation:

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\vec{J} = J_x + J_y + J_z \Rightarrow \vec{J}\cdot\hat{n} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & \sin\theta & 0 \\ \sin\theta & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} + \hbar\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\cos\theta \end{pmatrix}$$

$$= \hbar\begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & 0 & \frac{\sin\theta}{\sqrt{2}} \\ 0 & \frac{\sin\theta}{\sqrt{2}} & -\cos\theta \end{pmatrix}$$

Need to find the eigenvalues of this matrix:

$$\begin{vmatrix} \cos\theta - \lambda & \frac{\sin\theta}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & -\lambda & \frac{\sin\theta}{\sqrt{2}} \\ 0 & \frac{\sin\theta}{\sqrt{2}} & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)\left[\lambda(\cos\theta + \lambda) - \frac{\sin^2\theta}{2}\right] - \frac{\sin\theta}{\sqrt{2}}\left[-\frac{\sin\theta}{\sqrt{2}}(\cos\theta + \lambda)\right] + 0 = 0$$

$$\lambda\cos^2\theta + \cancel{\lambda^2\cos\theta} - \cancel{\frac{\sin^2\theta\cos\theta}{2}} - \cancel{\lambda^2\cos\theta} - \lambda^3 + \underline{\frac{\lambda\sin^2\theta}{2}} + \cancel{\frac{\sin^2\theta\cos\theta}{2}} + \underline{\frac{\lambda\sin^2\theta}{2}} = 0$$

$$-\lambda^3 + \underbrace{\lambda\cos^2\theta + \lambda\sin^2\theta}_{=\lambda} = 0 \Rightarrow \lambda^3 - \lambda = 0 \Rightarrow \lambda(\lambda^2-1) = 0$$

so $\lambda = 0, 1, -1$ hence the eigenvalues are $\boxed{0, \hbar, -\hbar}$

As for the probability of getting $+\hbar$ again, we first need the eigenfunction corresponding to $\lambda = +1$:

$$\begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & 0 & \frac{\sin\theta}{\sqrt{2}} \\ 0 & \frac{\sin\theta}{\sqrt{2}} & -\cos\theta \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 1 \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \Rightarrow \begin{aligned} &A\cos\theta + B\frac{\sin\theta}{\sqrt{2}} = A && (1) \\ &A\frac{\sin\theta}{\sqrt{2}} + C\frac{\sin\theta}{\sqrt{2}} = B && (2) \\ &B\frac{\sin\theta}{\sqrt{2}} - C\cos\theta = C && (3) \end{aligned}$$

From (1): $B\frac{\sin\theta}{\sqrt{2}} = A - A\cos\theta = A(1-\cos\theta) \Rightarrow B = A\frac{\sqrt{2}(1-\cos\theta)}{\sin\theta}$

(3): $B\frac{\sin\theta}{\sqrt{2}} = C + C\cos\theta = C(1+\cos\theta) \Rightarrow C = A\frac{(1-\cos\theta)}{(1+\cos\theta)}$

let $A = 1+\cos\theta \Rightarrow C = 1-\cos\theta$; $B = \sqrt{2}\frac{(1+\cos\theta)(1-\cos\theta)}{\sin\theta} = \sqrt{2}\frac{\overbrace{(1-\cos^2\theta)}^{=\sin^2\theta}}{\sin\theta} = \sqrt{2}\sin\theta$

so $N\begin{pmatrix} 1+\cos\theta \\ \sqrt{2}\sin\theta \\ 1-\cos\theta \end{pmatrix}$ is the un-normalized eigenfunction ($|e\rangle$)

$$N^2\left[(1+\cos\theta)^2 + 2\sin^2\theta + (1-\cos\theta)^2\right] = 1 \Rightarrow N^2\left[1 + \cancel{2\cos\theta} + \cos^2\theta + 2\sin^2\theta + 1 - \cancel{2\cos\theta} + \cos^2\theta\right] = 1$$

$$\Rightarrow N^2\left[2 + 2\overbrace{(\cos^2\theta + \sin^2\theta)}^{=1}\right] = 1 \Rightarrow N^2 \cdot 4 = 1 \Rightarrow N^2 = \frac{1}{4} \Rightarrow N = \frac{1}{2}$$

As for the probability: (where $|i\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ as $J_z = +\hbar$)

$$|\langle e|i\rangle|^2 = \left[\frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sqrt{2}\sin\theta & 1-\cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]^2 = \frac{1}{4}\underbrace{(1+\cos\theta)^2}_{(2\cos^2\theta/2)^2} = \boxed{\cos^4\theta/2}$$