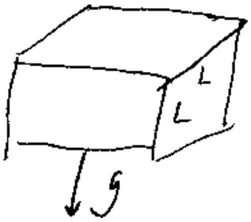
Problem 14 . 1998



$$z = \sum_r e^{-\beta E_r} \qquad E_r = \frac{p_r^2}{2m} + mgz$$

Box is large enough to have continous energy levels

$$Z = \iint e^{-\beta\left(\frac{p^2}{2m} + mgz\right)} d^3p\, d^3r$$

$$\overline{KE} = \frac{\int \frac{p^2}{2m} e^{-\beta\frac{p^2}{2m}} d^3p \int e^{-\beta mgz} d^3r}{\int e^{-\beta\frac{p^2}{2m}} d^3p \int e^{-\beta mgz} d^3r} = -\frac{\partial}{\partial\beta} \ln \int e^{-\beta\frac{p^2}{2m}} d^3p$$

$$= -\frac{\partial}{\partial\beta} \ln \left[\int_{-\infty}^{\infty} e^{-\beta\frac{p^2}{2m}} dp\right]^3 \quad \text{since} \quad \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$= -\frac{\partial}{\partial\beta} \ln \left[\pi^{1/2}\left(\frac{2m}{\beta}\right)^{1/2}\right]^3 = -3\frac{\partial}{\partial\beta}\left[\frac{1}{2}\ln\pi + \frac{1}{2}\ln(2m) - \frac{1}{2}\ln\beta\right]$$

$$= \frac{3}{2\beta} = \frac{3}{2}kT$$

$$\overline{PE} = \frac{\int e^{-\beta\frac{p^2}{2m}} d^3p \int mgz\, e^{-\beta mgz} d^3r}{Z} \Rightarrow -\frac{\partial}{\partial\beta} \ln \int_0^L e^{-\beta mgz} dz$$

$$= -\frac{\partial}{\partial\beta} \ln\left[\left(-\frac{1}{\beta mg}\right) e^{-\beta mgz}\Big|_0^L\right] = -\frac{\partial}{\partial\beta} \ln\left[\left(\frac{1}{\beta mg}\right)\left(e^0 - e^{-\beta mgL}\right)\right]$$

$$-\frac{\partial}{\partial\beta}\left[-\ln\beta - \ln(mg) + \ln\left(1 - e^{-\beta mgL}\right)\right]$$

$$= \frac{1}{\beta} - \frac{1}{(1-e^{-\beta mgL})} mgL\, e^{-\beta mgL} = \boxed{kT + mgL\left(\frac{1}{e^{\beta mgL} - 1}\right)}$$