(↑) Review completely

Fall 1998 #2

p63

Consider a sphere of paramagnetic material in an external applied magnetic field $\vec{B} = B_o \hat{z}$. Let the sphere have radius R_o and suppose that inside the sphere, the magnetization satisfies

$$\vec{M}(\vec{B}) = \frac{M_o \vec{B}}{B_c + B}$$

where $B = |\vec{B}|$, and where M_o and B_c are positive constants which obey $8\pi M_o / 3B_c < 1$. Calculate the field strength inside and outside the sphere.

(refer to Greiner pgs 229-231)

→ Start by considering a uniformly magnetized sphere without an applied field. Maxwell's equation $\nabla \times \vec{H} = \frac{4\pi}{c} J_f + \frac{1}{c}\frac{\partial D}{\partial t}$ tells us that $\nabla \times \vec{H} = 0$, allowing one to write $\vec{H} = -\nabla \Phi_M$ outside the sphere. The general solution for Φ_M is thus:

$$\Phi_M = \sum_{\ell=0}^{\infty} \left[\cancel{A_\ell r^\ell} + \frac{C_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta) \qquad r > R_o$$

Inside the sphere $\vec{H} = \vec{B}_i - 4\pi \vec{M}_i$ for $\vec{B}$ parallel to $\vec{M}$

Boundary conditions tell us that B_r, H_θ are continuous.

For magnetization along the $\hat{z}$ direction, we have inside the spher

$$\vec{B}_{in} = B_i \hat{z} = B_i(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$
$$\vec{H}_{in} = (B_i - 4\pi M_i)\hat{z} = (B_i - 4\pi M_i)(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

(using $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$; Arfken pg 12.

outside the sphere, $\vec{B} = \vec{H}$, so we have,

$$\vec{B}_{out} = \vec{H}_{out} = -\nabla\Phi_M = -\frac{\partial \Phi_m}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \Phi_M}{\partial \theta}\hat{\theta}$$

$$= \left[+(\ell+1) C_\ell r^{-(\ell+2)} P_\ell(\cos\theta)\right]\hat{r} - \left[C_\ell r^{-(\ell+2)} \frac{\partial}{\partial\theta} P_\ell(\cos\theta)\right]\hat{\theta}$$

In order to satisfy B_r continuous ℓ must equal 1, so

$$\vec{B}_{out} = \vec{H}_{out} = \left[+2Cr^{-3}\cos\theta\right]\hat{r} + \left[Cr^{-3}\sin\theta\right]\hat{\theta}$$

Applying the boundary conditions yields ⟹

$B_i = \frac{2C}{R_0^3}$ $\qquad + (B_i - 4\pi M_i) = -\frac{C}{R_0^3}$

solve for C

$$\frac{2C}{R_0^3} - 4\pi M_i = -\frac{C}{R_0^3}$$

$$C = \frac{4\pi}{3} M_i R_0^3$$

$$B_i = \frac{8\pi}{3} M_i$$

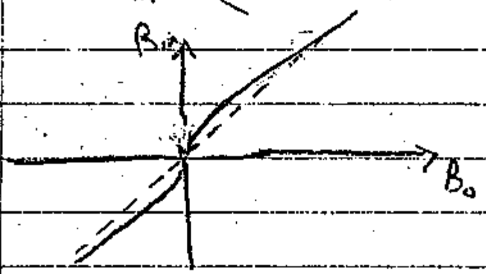
Now if the magnetization is due to an applied field, we get:

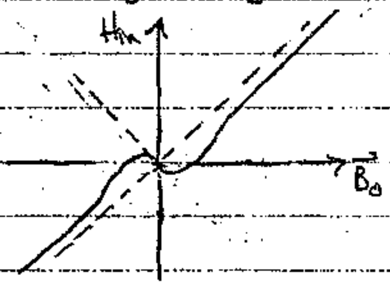
why? $\vec{B}_i = B_0 + \frac{8\pi}{3} M_i$ $\qquad H_i = -\frac{4\pi}{3} M_i + B_0$

thus our fields are

$$\boxed{\begin{aligned}
\vec{B}_{in} &= B_0\left(1 + \frac{8\pi M_0}{3(B_c+B_0)}\right)(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \\
\vec{H}_{in} &= B_0\left(1 - \frac{4\pi M_0}{3(B_c+B_0)}\right)(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \\
\vec{B}_{OUT} = \vec{H}_{out} &= \left[B_0 + \frac{8\pi}{3}\frac{M_0 R_0^3}{(B_c+B_0)r^3}\right]\cos\theta\,\hat{r} + \left[B_0 + \frac{4\pi}{3}\frac{M_0 R_0^3}{(B_c+B_0)r^3}\right]\sin\theta
\end{aligned}}$$

Sketch the field strength at the center of the sphere for $\vec{B}_{in} = (3B_0^2 + 3B_0B_c + 8\pi M_0 B_0)/(3(B_c+B_0))$, $\vec{H}_{in} = (3B_0^2 + 3B_0 B_c - 4\pi M_0 B_0)/(3(B_c$





Taking a closer look at H_{in}, one can rearrange to write

$$H_{in} = B_0\left(1 - \frac{4\pi M_0}{3(B_c+B_0)}\right) = B_0\left[1 - \underbrace{\left(\frac{8\pi M_0}{3B_c}\right)}_{>1}\underbrace{\left(\frac{1}{2(1+B_0/B_c)}\right)}_{\text{always } <1}\right]$$

for $\left(\frac{8\pi M_0}{3B_c}\right) > 1$ it takes a large applied field to get $\vec{H}$ to point parallel to the applied $\vec{B}$ field