Fall 1998 #4

Radiation from a charge distribution: In the radiation zone, the vector potential from a current oscillating at frequency ω is:

$$\vec{A}_\omega(\vec{r}) = \frac{e^{i(kr-\omega t)}}{cr}\int \vec{J}_\omega(\vec{r}\,')\, e^{-i\vec{k}\cdot\vec{r}\,'}\, d^3\vec{r}\,'$$

where $\omega = kc$, $\vec{k} = k\hat{r}$, $\hat{r} = \vec{r}/r$.

(a) Suppose $J_\omega(\vec{r}) = J_0(\omega)\,\hat{x}\, e^{-r^2/a^2}$, where $\hat{x}$ is the unit vector in the x direction. Calculate the electric and magnetic fields in the radiation zone.

expand $e^{-i\vec{k}\cdot\vec{r}\,'} = 1 - i\vec{k}\cdot\vec{r}\,' + \frac{(\vec{k}\cdot\vec{r}\,')^2}{2} - \ldots$

in the radiation zone, keep only the first term.

$$\vec{A}_\omega(\vec{r}) = \frac{e^{i(kr-\omega t)}}{cr}\int J_0(\omega)\,\hat{x}\, e^{-r'^2/a^2}\, r'^2 \sin\theta\, dr'\, d\theta\, d\phi$$

$$= \frac{4\pi e^{i(kr-\omega t)}}{cr} J_0(\omega)\,\hat{x}\int_0^\infty e^{-\frac{r'^2}{a^2}}\, r'^2\, dr$$

$$= \frac{4\pi e^{i(kr-\omega t)}}{cr} J_0(\omega)\,\frac{1}{4}\sqrt{\pi}\, a^3$$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{i(kr-\omega t)}}{cr} J_0(\omega)\,\pi^{3/2} a^3\,\hat{x}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \text{(Griffiths pg 314)}$$

$$= -\frac{1}{c}(-i\omega)\frac{e^{i(kr-\omega t)}}{cr} J_0(\omega)\,\pi^{3/2} a^3\,\hat{x}$$

$$= \frac{i\omega}{c^2 r}\pi^{3/2} a^3\,\hat{x}\left[\cos(kr-\omega t) + i\sin(kr-\omega t)\right]$$

taking real part of $\vec{E}$ yields

$$\boxed{\vec{E}(r) = \frac{-\omega}{c^2 r}\pi^{3/2} a^3 \sin(kr-\omega t)\,\hat{x}}$$

$$\vec{B} = \nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & 0 & 0 \end{vmatrix} = -\hat{y}\left(-\frac{\partial}{\partial z}A_x\right) + \hat{z}\left(-\frac{\partial}{\partial y}A_y\right)$$

$\Rightarrow$

$$\frac{\partial}{\partial z} A_x = \left(\frac{\partial}{\partial r} A_x\right)\left(\frac{\partial r}{\partial z}\right) \quad ; \quad r = (x^2+y^2+z^2)^{1/2} \quad \Rightarrow \frac{dr}{dz} = \frac{1}{2}(r^{-1/2}$$

$$= \frac{J_0(\omega)}{c}\pi^{3/2} a^3 \left[\frac{-e^{i(kr-\omega t)}}{r^2} + \frac{rik\, e^{i(kr-\omega t)}}{r^2}\right] \frac{z}{\sqrt{r}}$$

$$-\frac{\partial}{\partial y} A_x = \left(\frac{\partial}{\partial r} A_x\right)\left(\frac{\partial r}{\partial y}\right)$$

$$= -\frac{J_0(\omega)\pi^{3/2} a^3}{c}\left[\frac{-e^{i(kr-\omega t)}}{r^2} + \frac{rik\, e^{i(kr-\omega t)}}{r^2}\right]\frac{y}{\sqrt{r}}$$

Thus
$$\boxed{\vec{B} = \frac{J_0(\omega)\pi^{3/2} a^3}{c r^2 \sqrt{r}} e^{i(kr-\omega t)}\left[(ikr-1) z\hat{y} - (ikr-1) y\hat{z}\right]}$$

c) Give an expression for the total power radiated.

$$P = \int \vec{S}\cdot d\vec{a} \qquad S = \frac{c}{4\pi}(E^* \times B)$$

$$\boxed{P = \frac{c}{4\pi}\left\{J_0(\omega)\pi^{3/2} a^3\right\}^2 \left(-\frac{i\omega}{c^3}\right)\int \frac{1}{r^3\sqrt{r}}\left[(ikr-1) z\hat{z} + (ikr-1) y\hat{y}\right] r^2 \sin\theta\, d}$$