Fall 1998 Solution 4

Andrew Collette

August 12, 2005

1 Approximations

The expression given is an approximation of the general formula for $\mathbf{A}$ (Jackson 9.3):

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathbf{J}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r' \tag{1}$$

where we have expanded $k|\mathbf{r}-\mathbf{r}'| \approx kr - \mathbf{k}\cdot\mathbf{r}'$ in the exponential and $|\mathbf{r}-\mathbf{r}'| \approx r$ in the denominator. Both these expansions are legal for $r >> \lambda >> a$ where a is the scale of the system. This is the far-field approximation. The integration over r' in the given expression makes no reference to the magnitude of r; it is concerned only with the structure of the source. Without knowing how the size of the source compares to a wavelength, it is technically incorrect to make further approximations. For example, if we know that $a << \lambda$, we could approximate the source as a dipole, considerably simplifying the problem by replacing the exponential factor $e^{-i\mathbf{k}\cdot\mathbf{r}'}$ by 1. The other set of solutions does this. Because the problem does not specify the relative size of the source, I will solve the problem without making further approximations.

2 Solution

2.1 Vector Potential

First, we have to find a simple expression for $\mathbf{A}(\mathbf{r})$. Substitute the given $\mathbf{J}(w)$:

$$\mathbf{A}_\omega(\mathbf{r},t) = \mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\int \mathbf{J}_0(\omega)e^{\frac{-r'^2}{a^2}}e^{-i\mathbf{k}\cdot\mathbf{r}'}d^3r' \tag{2}$$

Now, orient the z' axis to face along the direction of $\mathbf{r}$. This is legal, as nothing in the r' integral makes reference to the r coordinate system, except the dot product $\mathbf{k}\cdot\mathbf{r}'$. Since $\mathbf{k}$ points along $\mathbf{r}$, we can replace the dot product explicitly by

$$\mathbf{k}\cdot\mathbf{r}' = kr'\cos\theta' \tag{3}$$

and proceed to break up the integral and integrate:

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\mathbf{J}_0(\omega)\int e^{\frac{-r'^2}{a^2}}e^{-ikr'\cos\theta'}r'^2\sin\theta' dr' d\phi' d\theta' \tag{4}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\mathbf{J}_0(\omega)\int_0^\infty dr' r'^2 e^{\frac{-r'^2}{a^2}} \int_0^{2\pi} d\phi' \int_0^\pi e^{-ikr'\cos\theta'}\sin\theta' d\theta' \tag{5}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}2\pi\mathbf{J}_0(\omega)\int_0^\infty dr' r'^2 e^{\frac{-r'^2}{a^2}} \int_0^\pi e^{-ikr'\cos\theta'}\sin\theta' d\theta' \tag{6}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}2\pi\mathbf{J}_0(\omega)\int_0^\infty dr' r'^2 e^{\frac{-r'^2}{a^2}} (\frac{-1}{-ikr'})[e^{-ikr'\cos\theta'}]_0^\pi \tag{7}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\frac{2\pi}{ik}\mathbf{J}_0(\omega)\int_0^\infty dr' r' e^{\frac{-r'^2}{a^2}} [e^{ikr'} - e^{-ikr'}] \tag{8}$$

But $e^{ikr'} - e^{-ikr'} = 2i\sin ikr'$, so:

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\frac{4\pi}{k}\mathbf{J}_0(\omega)\int_0^\infty dr' r' e^{\frac{-r'^2}{a^2}} \sin kr' \tag{9}$$

Integrate by parts:

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\frac{4\pi}{k}\mathbf{J}_0(\omega)\frac{-a^2}{2}\int_0^\infty dr' (\frac{-2r'}{a^2}) e^{\frac{-r'^2}{a^2}} \sin kr' \tag{10}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\frac{4\pi}{k}\mathbf{J}_0(\omega)\frac{-a^2}{2}([e^{\frac{-r'^2}{a^2}}\sin kr']_0^\infty - k\int_0^\infty e^{\frac{-r'^2}{a^2}}\cos kr' dr') \tag{11}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\frac{4\pi}{k}\mathbf{J}_0(\omega)\frac{-a^2}{2}(0 - k(\frac{1}{2})\sqrt{\pi}ae^{\frac{-k^2a^2}{4}}) \tag{12}$$

$$\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr}\mathbf{J}_0(\omega)\pi^{3/2}a^3 e^{\frac{-k^2a^2}{4}} \tag{13}$$

Note that in the limit $ka << 1$ (i.e. $a << \lambda$) the exponential goes to 1, and this reduces to the dipole approximation solution. The extra factor changes the magnitude of $\mathbf{A}$ but does not introduce any new angular effects. So the dipole approximation is not necessarily bad, especially if you're in a rush. The qualitative behavior will be the same in this case.

2.2 Electric Field

Now that we have the vector potential, we can find the electric field very easily:

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} \tag{14}$$

$$\mathbf{E} = -i\omega t\frac{e^{i(kr-\omega t)}}{c^2 r}J_0(\omega)\pi^{3/2}a^3 e^{\frac{-k^2a^2}{4}}\hat{x} \tag{15}$$

2.3 Magnetic Field

The magnetic field is just the curl of the vector potential, which is simplified because $A_y = A_z = 0$:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{dA_x}{dz}\hat{y} - \frac{dA_x}{dy}\hat{z} \tag{16}$$

Since we have $\mathbf{A}$ in terms of r and t, we need to translate:

$$\mathbf{B} = \frac{dA_x}{dr}\frac{dr}{dz}\hat{y} - \frac{dA_x}{dr}\frac{dr}{dy}\hat{z} \tag{17}$$

$$\frac{dr}{dz} = \frac{z}{\sqrt{x^2+y^2+z^2}}; \quad \frac{dr}{dy} = \frac{y}{\sqrt{x^2+y^2+z^2}}; \tag{18}$$

By inspection,

$$\frac{dA_x}{dr} = (ik - \frac{1}{r})A_x \tag{19}$$

so

$$\mathbf{B} = (ik - \frac{1}{r})A_x(\frac{z}{r}\hat{y} - \frac{y}{r}\hat{z}) \tag{20}$$

$$\mathbf{B} = (ikr-1)\frac{e^{i(kr-\omega t)}}{cr^3}J_0(\omega)\pi^{3/2}a^3e^{-\frac{k^2a^2}{4}}(z\hat{y} - y\hat{z}) \tag{21}$$

2.4 Total power radiated

Since we now have explicit forms for $\mathbf{E}$ and $\mathbf{B}$, one way to find the total power is to find the Poynting vector and integrate it over the sphere. This is perfectly valid; see the other set of solutions for an implementation. There is a shortcut, however. Treat the system as a dipole for the moment. The dipole moment is (Jackson 9.14, 9.17):

$$\mathbf{p} = -\frac{1}{i\omega}\int \mathbf{J}(\mathbf{r})d^3r \tag{22}$$

$$\mathbf{p} = -\frac{4\pi}{i\omega}\mathbf{J_0}\int_0^\infty r^2e^{-\frac{r^2}{a^2}} = -\frac{4\pi}{i\omega}\mathbf{J_0}\frac{\sqrt{\pi}}{4}a^3 = -\frac{\pi^{3/2}a^3}{i\omega}\mathbf{J_0} \tag{23}$$

If we were to find the Poynting vector using our $\mathbf{E}$ and $\mathbf{B}$, we would do

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E}^* \times \mathbf{B}) = \frac{c}{4\pi}(e^{-\frac{k^2a^2}{4}})^2(\mathbf{E}^*_{dipole} \times \mathbf{B}_{dipole}) = (e^{-\frac{k^2a^2}{4}})^2\mathbf{S}_{dipole} \tag{24}$$

So we can say

$$P_{total} = e^{-\frac{k^2a^2}{2}}P_{dipole} \tag{25}$$

But we know the power radiated by a dipole; it's

$$P = \frac{ck^4}{3}|\mathbf{p}^2| \tag{26}$$

so in our case

$$P = e^{-\frac{k^2a^2}{2}}\frac{ck^4}{3}\frac{\pi^3a^6}{\omega^2}J_0^2 = e^{-\frac{k^2a^2}{2}}\frac{\pi^3}{3}\frac{k^2}{c}a^6J_0^2 \tag{27}$$