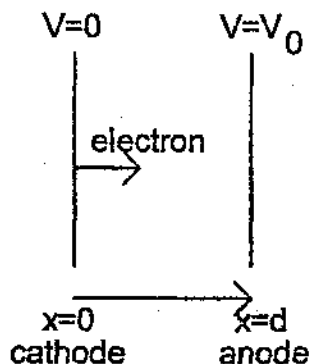


Consider an idealized parallel plate diode, as shown. At equilibrium, electrons are emitted in large numbers with negligible velocity from the cathode (at voltage $V=0$) and are attracted to the anode (held at constant $V=V_0$). Find the voltage (potential) $V(x)$ and the current density $J(x)$ between the plates at equilibrium, in terms of the electron mass m , charge magnitude e , V_0 , d , and x . Hint: At the appropriate place in the solution, assume that the functional form of V is $V(x) \propto x^y$ for some (real) y .

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The continuity equation tells us that

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad (i)$$

Also, Poisson's equation states

$$\nabla \cdot \vec{E} = 4\pi\rho$$

Or, in terms of potential,

$$\nabla^2 V = -4\pi\rho \quad (ii)$$

Furthermore, we know that

$$\vec{J} = \rho\vec{v} \quad (iii)$$

Where: $\vec{v} = \text{velocity}$

Combining equations (ii), (iii) yields

$$\nabla^2 V = -4\pi \frac{\vec{J}}{v}$$

For the one dimensional case

$$\frac{d^2 V}{dx^2} = -4\pi \frac{J}{v} \quad (iv)$$

Next, find the velocity as a function of x .

Start with

$$\vec{F} = -e\vec{E} = e \frac{dV}{dx} = ma$$

This can be rewritten as

$$e \frac{dV}{dx} = m \frac{dv}{dt} = m \frac{dv}{dx} \frac{dx}{dt}$$

Recognize that $\frac{dx}{dt} = v$, and rearrange to yield

$$v dv = \frac{e}{m} \frac{dV}{dx} dx$$

Integration and the condition that initial velocity is zero yield:

$$\frac{1}{2} v^2(x) = \frac{e}{m} V(x)$$

$$v(x) = \sqrt{\frac{2e}{m} V(x)}$$

From the hint, assume that $V(x) = Ax^y$

From the condition $V(x=d) = Ad^y = V_0$, one finds that $A = \frac{V_0}{d^y}$, so

$$v(x) = \sqrt{\frac{2e}{m} Ax^y}$$

From equation (iv), one can find

$$-4\pi \frac{J}{v(x)} = \frac{d^2}{dx^2} (Ax^y) = \frac{d}{dx} (Ayx^{(y-1)}) = Ay(y-1)x^{(y-2)}$$

$$J(x) = -\frac{1}{4\pi} \left(\frac{m}{2eA} \right)^{1/2} Ay(y-1)x^{(y-2+y/2)}$$

However, from (i) it is possible to see that $J(x)$ must be constant, otherwise charge would be accumulating. In order to satisfy this condition, it is necessary for the exponent of x to be zero.

$$x^{(3y/2-2)} = 1$$

$$\frac{3}{2}y - 2 = 0$$

$$y = \frac{4}{3}$$

The final result is thus,

$$V(x) = \frac{V_0}{d^{4/3}} x^{4/3}$$

$$J(x) = -\frac{1}{9\pi} \left(\frac{mV_0}{2ed^{4/3}} \right)^{1/2}$$