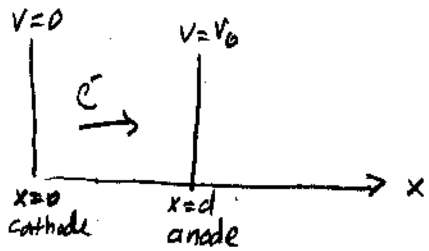
Fall 1998 #5 (p1 of 2)

Consider an idealized parallel plate diode, as shown. At equilibrium, electrons are emitted in large numbers with negligible velocity from the cathode (at voltage $V=0$) and are attracted to the anode (held at constant $V=V_0$). Find the voltage (potential) $V(x)$ and the current density $J(x)$ between the plates at equilibrium, in terms of the electron mass m, charge magnitude e, V_0, d, and x.
Hint: At the appropriate place in the solution, assume the functional form of V is $V(x) \sim x^y$ for some (real) y.



→ see solution to Morales Part A HW set 4 add prob #2 & Griffiths E&M 2.48

Start with Poisson's equation: $\nabla \cdot \vec{E} = 4\pi\rho$, $\vec{E} = -\nabla V \Rightarrow \nabla^2 V = -4\pi\rho$
from Griffiths' eq 5.26, we have $\vec{J} = \rho \vec{v}$

$$\Rightarrow \boxed{\frac{\partial^2 V(x)}{\partial x^2} = \frac{-4\pi J(x)}{v(x)}} \quad (1)$$

in 1-D

Now, we want to find $V(x)$ and $J(x)$. In order to accomplish this, we need to find $v(x)$. To do that, let's use what we know about the relationship between work and the change in kinetic energy. That is,

$$W = \int_{x=0}^{x=d} \vec{F}_e \cdot d\vec{x} = \frac{1}{2} m_e v(x)^2 - 0 \quad , \vec{F}_e = -e\left[-\frac{\partial V(x)}{\partial x}\right] = e\frac{\partial V(x)}{\partial x}$$

So, we have

$$\frac{1}{2} m_e v(x)^2 = e\left[V(x=d) - V(x=0)\right] = eV_0 - 0 \quad \Rightarrow \boxed{v(x) = \sqrt{\frac{2eV(x)}{m}}} \quad (2)$$

Now, employ the hint. Let $V(x) \sim x^y \Rightarrow V(x) = Ax^y$, what is A?
we can find A from the boundary condition at $x=d$. That is,

$$V(d) = V_0 = Ad^y \quad \Rightarrow A = \frac{V_0}{d^y}$$

Thus,

$$\boxed{V(x) = V_0\left(\frac{x}{d}\right)^y} \quad (3)$$

substituting eq (3) into eq (2) yields

$$\boxed{v(x) = \sqrt{\frac{2eV_0}{m}\left(\frac{x}{d}\right)^y}} \quad (4)$$

substituting eq (4) into eq (1) yields

$$\frac{\partial^2}{\partial x^2}\left[V_0\left(\frac{x}{d}\right)^y\right] = \frac{-4\pi J(x)}{\sqrt{\frac{2eV_0}{m}\left(\frac{x}{d}\right)^y}}$$

$$\Rightarrow \; -4\pi J(x) = \sqrt{\frac{2eV_0}{m}\left(\frac{x}{d}\right)^y}\left[\frac{V_0}{d^y}\right] y(y-1)\,x^{y-2}$$

$$\Rightarrow \boxed{J(x) = -\frac{1}{4\pi}\sqrt{\frac{2e}{m}}\;\frac{V_0^{3/2}}{d^{3y/2}}\;y(y-1)\,x^{3y/2-2}} \qquad (5)$$

From the continuity equation $\nabla\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$, we know that $J(x)$ must be constant wrt x or else charge would be accumulating and not in motion. Thus, the power of x must vanish. That is,

$$\frac{3y}{2} - 2 = 0 \qquad \Rightarrow \; y = \frac{4}{3}$$

substituting the value of y into eq (3) yields

$$\boxed{V(x) = V_0\left(\frac{x}{d}\right)^{4/3}}$$

and into eq (5) yields

$$J(x) = -\frac{1}{4\pi}\sqrt{\frac{2e}{m}}\;\frac{V_0^{3/2}}{d^2}\;\frac{4}{3}\left(\frac{1}{3}\right)$$

$$\therefore \; \boxed{J(x) = -\frac{V_0^{3/2}}{9\pi d^2}\sqrt{\frac{2e}{m}}}$$

→ this answer for $J(x)$ is different than the solutions... ???

but the same as the solution in Morales' class:

$$\frac{I}{A} = \frac{4}{9}\sqrt{\frac{2e}{m_e}}\;\frac{\epsilon_0}{d^2}\,V_1^{3/2} = \frac{V_0^{3/2}}{9\pi d^2}\sqrt{\frac{2e}{m}} \quad , \; \epsilon_0 \to \frac{1}{4\pi}$$