The equations of state for an ideal gas are:

$$PV = NkT \quad ; \quad U = \frac{3}{2}NkT$$

where N is the number of particles. Derive the Helmholtz free energy, $F(N,V,T)$ of the ideal gas. From this quantity, find the entropy as a function of T, N, V.

→ (refer to Reif pgs 238-245)

Start with the partition function of the gas: $Z = \frac{(z_1)^N}{N!}$

where z_1 is the partition function of a single molecule.

$$z_1 = \iint e^{-\beta E(\vec{r},\vec{p})}\, d^3r\, d^3p$$

for an ideal gas, the energy of a molecule is purely kinetic, so $E = \frac{1}{2m}\vec{p}^{\,2}$, leaving:

$$z_1 = \iint e^{-\beta \frac{p^2}{2m}} d^3r\, d^3p$$

Now the integral $\int d^3r$ gives the volume, while the integral $\int d^3p$ can be written as:

$$\int e^{-\beta\frac{p^2}{2m}} d^3p = \underbrace{\int_{-\infty}^{\infty} e^{-\beta\frac{p_x^2}{2m}} dp_x}_{=\sqrt{\frac{\pi 2m}{\beta}}} \underbrace{\int_{-\infty}^{\infty} e^{-\beta\frac{p_y^2}{2m}} dp_y}_{=\sqrt{\frac{\pi 2m}{\beta}}} \underbrace{\int_{-\infty}^{\infty} e^{-\beta\frac{p_z^2}{2m}} dp_z}_{=\sqrt{\frac{\pi 2m}{\beta}}}$$

So $z_1 = V(2\pi mkT)^{3/2}$

Note: if one treats phase space as discrete, an extra factor $\frac{1}{h_0^3}$ appears in the expression for z_1. In a quantum treatment this factor is $\frac{1}{h^3}$ where h equals Planck's constant.

$$Z = \frac{(z_1)^N}{N!} = \left(\frac{1}{N!}\right) V^N (2\pi mkT)^{3N/2}$$

$$F = -kT\ln Z = -kT\left[N\ln V + \frac{3N}{2}\ln(2\pi mkT) - \ln(N!)\right]$$

now use $\ln(N!) \approx N\ln N - N$ to write:

$$F = -kT\left[N\ln V + \frac{3N}{2}\ln(2\pi mkT) - N\ln N + N\right]$$

$$\boxed{F = -NkT\left[\ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln(2\pi mkT) + 1\right]}$$

⇒

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

$$S = Nk\ln\left(\frac{V}{N}\right) + \frac{3Nk}{2}\ln(2\pi mkT) + \frac{3}{2}\frac{Nk\cancel{T}\,\cancel{(2\pi mk)}}{\cancel{(2\pi mkT)}} + Nk$$

$$\boxed{S(T,N,V) = Nk\ln\left(\frac{V}{N}\right) + \frac{3Nk}{2}\ln(2\pi mkT) + \frac{5}{2}Nk}$$

As a side note, one can verify the equations of state using

$$\bar{p} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V} = kT\left(\frac{1}{Z}\right)\underbrace{\left(N V^{N-1}\right)\left(\frac{1}{N!}(2\pi mkT)^{3/2}\right)}_{=\frac{Z}{V}}$$

$$\boxed{\bar{p} = \frac{NkT}{V}}$$

$$U = \bar{E} = -\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\ln\left[\left(\frac{1}{N!}\right)V^N\left(\frac{2\pi m}{\beta}\right)^{3N/2}\right]$$

$$U = -\frac{1}{Z}\left(-\frac{3N}{2}\right)\underbrace{\beta^{\left(\frac{3N}{2}-1\right)}\left(\frac{1}{N!}\right)V\ (2\pi m)^{3/2}}_{=Z/\beta}$$

$$U = \frac{3}{2}\frac{N}{\beta}$$

$$\boxed{U = \frac{3}{2}NkT}$$