Fall 1998 #7

Consider a collection of N particles, each of which can occupy one of two single-particle states, with energies ϵ_0 and ϵ_1, respectively.

(a). In terms of N, U, ϵ_0, and ϵ_1, what is the entropy of this system?

$\rightarrow$ Start with: $\quad N = N_0 + N_1 \quad ; \quad U = N_0\epsilon_0 + N_1\epsilon_1$

solve for N_0, N_1

$$U = (N - N_1)\epsilon_0 + N_1\epsilon_1 \qquad\qquad U = N_0\epsilon_0 + (N - N_0)\epsilon_1$$

$$U = N\epsilon_0 + N_1(\epsilon_1 - \epsilon_0) \qquad\qquad U = N\epsilon_1 + N_0(\epsilon_0 - \epsilon_1)$$

$$N_1 = \frac{U - N\epsilon_0}{\epsilon_1 - \epsilon_0} \qquad\qquad N_0 = \frac{N\epsilon_1 - U}{\epsilon_1 - \epsilon_0}$$

$$S = k\ln\Omega \quad , \quad \Omega = \frac{N!}{N_0!\,N_1!}$$

$$\boxed{S = k\ln\left[\frac{N!}{\left(\frac{N\epsilon_1 - U}{\epsilon_1 + \epsilon_0}\right)!\left(\frac{U - N\epsilon_0}{\epsilon_1 - \epsilon_0}\right)!}\right]}$$

b) Rewrite the expression in the form appropriate for $N \gg 1$.

$$\frac{S}{k} = \ln N! - \ln\left(\frac{N\epsilon_1 - U}{\epsilon_1 - \epsilon_0}\right)! - \ln\left(\frac{U - N\epsilon_0}{\epsilon_1 - \epsilon_0}\right)!$$

$$\boxed{\frac{S}{k} = N\ln N - N - \left\{\left(\frac{N\epsilon_1 - U}{\epsilon_1 - \epsilon_0}\right)\ln\left(\frac{N\epsilon_1 - U}{\epsilon_1 - \epsilon_0}\right) - \left(\frac{N\epsilon_1 - U}{\epsilon_1 - \epsilon_0}\right)\right\} - \left\{\left(\frac{U - N\epsilon_0}{\epsilon_1 - \epsilon_0}\right)\ln\left(\frac{U - N\epsilon_0}{\epsilon_1 - \epsilon_0}\right) - \left(\frac{U - N\epsilon_0}{\epsilon_1 - \epsilon_0}\right)\right.}$$

c) Obtain an expression for the temperature $T(U,N)$ of this system.

$$\beta = \frac{\partial\ln\Omega}{\partial U} \quad \Rightarrow \quad \frac{1}{T} = \frac{\partial S}{\partial U}$$

$$\frac{\partial}{\partial U}\ln\Omega = \frac{\partial}{\partial U}\left[\ln N! - \ln N_0! - \ln N_1!\right]$$

use Stirling's formula $\Rightarrow$

$$\frac{\partial}{\partial U}\ln\Omega = \frac{\partial}{\partial U}\left\{N\ln N - N - N_0\ln N_0 + N_0 - N_1\ln N_1 + N_1\right\}$$

differentiate using chain rule

$$\frac{\partial}{\partial U}\ln\Omega = -\frac{N_0}{N_0}\,\frac{\partial N_0}{\partial U} - \ln N_0\left(\frac{\partial N_0}{\partial U}\right) + \frac{\partial N_0}{\partial U} - \frac{N_1}{N_1}\,\frac{\partial N_1}{\partial U} - \ln N_1\left(\frac{\partial N_1}{\partial U}\right) + \frac{\partial N_1}{\partial U}$$

now $\frac{\partial N_0}{\partial U} = \frac{-1}{\epsilon_1-\epsilon_0}$, $\frac{\partial N_1}{\partial U} = \frac{1}{\epsilon_1-\epsilon_0}$

so

$$\frac{\partial}{\partial U}\ln\Omega = \cancel{-1\left(\frac{-1}{\epsilon_1-\epsilon_0}\right)} - \left(\frac{-1}{\epsilon_1-\epsilon_0}\right)\ln\underbrace{\left(\frac{N\epsilon_1-U}{\epsilon_1-\epsilon_0}\right)}_{=N_0} + \cancel{\left(\frac{-1}{\epsilon_1-\epsilon_0}\right)} - \cancel{1\left(\frac{1}{\epsilon_1-\epsilon_0}\right)} - \left(\frac{1}{\epsilon_1-\epsilon_0}\right)\ln\underbrace{\left(\frac{U-N\epsilon_0}{\epsilon_1-\epsilon_0}\right)}_{=N_1} + (\epsilon$$

$$= \left(\ln\left(\frac{N\epsilon_1-U}{\epsilon_1-\epsilon_0}\right)\left(\frac{1}{\epsilon_1-\epsilon_0}\right) - \ln\left(\frac{U-N\epsilon_0}{\epsilon_1-\epsilon_0}\right)\left(\frac{1}{\epsilon_1-\epsilon_0}\right)\right)$$

$$\frac{1}{kT} = \frac{+1}{\epsilon_1-\epsilon_0}\left[\ln\left(\frac{N\epsilon_1-U}{U-N\epsilon_0}\right)\right]$$

$$\boxed{T = \frac{\epsilon_1-\epsilon_0}{k}\left[\ln\left(\frac{N\epsilon_1-U}{U-N\epsilon_0}\right)\right]^{-1}}$$