Fall '98 #7

Consider a collection of N particles, each of which can occupy one of two single-particle states, w/ energies $\mathcal{E}_0$ & $\mathcal{E}_1$, respectively

a.) In terms of N, U (the total internal energy of the collection of particles), $\mathcal{E}_0$ and $\mathcal{E}_1$, what is the entropy of this system?

b.) Rewrite your expression in the form appropriate for $N \gg 1$. (Hint: $\ln(N!) \approx N\ln N - N$.)

c.) Obtain an expression for the temperature $T(U,N)$ of this system. What unusual property does this temperature have? Comment on it.

Will use: $\Omega(n) = \dfrac{N!}{(N-n)!\,n!}$, $S = k\ln\Omega$

a.) $U = N_0\mathcal{E}_0 + N_1\mathcal{E}_1$

$U = N\mathcal{E}_0 + N_1(-\mathcal{E}_0 + \mathcal{E}_1)$ $\qquad$ $U = N_0\mathcal{E}_0 + N\mathcal{E}_1 - N_0\mathcal{E}_1$

$\Rightarrow N_1 = \dfrac{U - N\mathcal{E}_0}{\mathcal{E}_1 - \mathcal{E}_0}$ $\qquad$ $\Rightarrow N_0 = \dfrac{U - N\mathcal{E}_1}{\mathcal{E}_0 - \mathcal{E}_1}$

$$\Rightarrow \Omega = \frac{N!}{N_1!\,N_0!} = \frac{N!}{\left(\frac{U-N\mathcal{E}_0}{\mathcal{E}_1-\mathcal{E}_0}\right)!\left(\frac{U-N\mathcal{E}_1}{\mathcal{E}_0-\mathcal{E}_1}\right)!}$$

& $S = k\ln\Omega$

b.) given $\ln N! \cong N\ln N - N$

N, N_0, N_1 large

$$\frac{S}{k} = \ln N! - \ln\left(\frac{U-N\mathcal{E}_0}{\mathcal{E}_1-\mathcal{E}_0}\right)! - \ln\left(\frac{U-N\mathcal{E}_1}{\mathcal{E}_0-\mathcal{E}_1}\right)!$$

$$\cong N\ln N - N - \frac{U-N\mathcal{E}_0}{\mathcal{E}_1-\mathcal{E}_0}\ln\left[\frac{U-N\mathcal{E}_0}{\mathcal{E}_1-\mathcal{E}_0}\right] + \cancel{\frac{U-N\mathcal{E}_0}{\mathcal{E}_1-\mathcal{E}_0}}$$
$$- \frac{U-N\mathcal{E}_1}{\mathcal{E}_0-\mathcal{E}_1}\ln\left[\frac{U-N\mathcal{E}_1}{\mathcal{E}_0-\mathcal{E}_1}\right] + \cancel{\frac{U-N\mathcal{E}_1}{\mathcal{E}_0-\mathcal{E}_1}}$$

$$= N\ln N - \cancel{N} - \frac{U-N\varepsilon_0}{\varepsilon_1-\varepsilon_0}\ln\left[\frac{U-N\varepsilon_0}{\varepsilon_1-\varepsilon_0}\right] - \frac{U-N\varepsilon_1}{\varepsilon_0-\varepsilon_1}\ln\left[\frac{U-N\varepsilon_1}{\varepsilon_0-\varepsilon_1}\right] \cancel{+ N\frac{(-\varepsilon_0+\varepsilon_1)}{\varepsilon_0-\varepsilon_1}}$$

$$\Rightarrow S = kN\ln N - k\frac{U-N\varepsilon_0}{\varepsilon_1-\varepsilon_0}\ln\left[\frac{U-N\varepsilon_0}{\varepsilon_1-\varepsilon_0}\right] - k\frac{U-N\varepsilon_1}{\varepsilon_0-\varepsilon_1}\ln\left[\frac{U-N\varepsilon_1}{\varepsilon_0-\varepsilon_1}\right]$$

c.) $dU = TdS - pdV - \mu dN$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_N \quad \text{or} \quad \frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N$$

$$\frac{k}{T} = \frac{1}{\varepsilon_1-\varepsilon_0}\ln\left[\frac{N\varepsilon_1-U}{U-N\varepsilon_0}\right]$$

$$\Rightarrow T = \frac{\varepsilon_1-\varepsilon_0}{k}\ln\left[\frac{N\varepsilon_1-U}{U-N\varepsilon_0}\right]^{-1}$$

→ 2 odd things w/ Temp. here:

1) $\ln 1 = 0$, so $U - N\varepsilon_0 = N\varepsilon_1 - U$

$$\downarrow$$

$$N = \frac{2U}{\varepsilon_0+\varepsilon_1} = \frac{2(N_0\varepsilon_0 + N_1\varepsilon_1)}{\varepsilon_0+\varepsilon_1}$$

→ if N_0, N_1 ≈ evenly distributed ($N_0 \approx N_1$) w/ $N \approx 2N$

⇒ Then $T = 0$

2) $T = 0$ also if $\varepsilon_1 = \varepsilon_0$

unphysical!