p75

Fall 1998 #9

An electron is injected into a region where there is a constant magnetic field $\vec{B}$ along the z-axis. At t=0, the direction of the electron's momentum is along the x-axis, and it is completely polarized so that its spin is along the direction of the beam. Let θ be the angle between the electron's momentum and the expectation of its spin. What is $\theta(t)$?

The Hamiltonian for the system is:

$$H = -\vec{\mu}\cdot\vec{B} = g\mu_0 \frac{\vec{S}}{\hbar}\cdot\vec{B} = g\mu_0 \frac{B}{2}\sigma_z$$

The initial state corresponds to spin $\frac{\hbar}{2}\hat{x}$, finding the corresponding eigenvector yields:

$$\left(\frac{\hbar}{2}\sigma_x - \frac{\hbar}{2}I\right)\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Solving for a, b and normalizing gives the initial state vector:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Thus, $|\Psi(t)\rangle$ is given by:

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-igB\mu_0 t/2\hbar}\\e^{+igB\mu_0 t/2\hbar}\end{pmatrix}$$

Now one can calculate $\langle\vec{S}\rangle = \langle S_x\rangle\hat{x} + \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}$

Define: $\quad e^+ = e^{+igB\mu_0 t/2\hbar} \quad\quad e^- = e^{-igB\mu_0 t/2\hbar}$

Then, for $\langle S_x\rangle$:

$$\langle S_x\rangle = \frac{\hbar}{4}\begin{pmatrix}e^+ & e^-\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}e^-\\e^+\end{pmatrix} = \frac{\hbar}{4}\left[(e^+)^2 + (e^-)^2\right]$$

$$\langle S_x\rangle = \frac{\hbar}{4}\left(\cos(gB\mu_0 t/\hbar) + i\sin(gB\mu_0 t/\hbar) + \cos(gB\mu_0 t/\hbar) - i\sin(gB\mu_0 t/\hbar)\right)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\cos(gB\mu_0 t/\hbar)$$

Similarly, for $\langle S_y\rangle$:

$$\langle S_y\rangle = \frac{\hbar}{4}\begin{pmatrix}e^+ & e^-\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}e^-\\e^+\end{pmatrix} = \frac{\hbar}{4}\left[-i(e^+)^2 + i(e^-)^2\right]$$

$$\langle S_y\rangle = \frac{\hbar}{4}\left(-i\cos(gB\mu_0 t/\hbar) + i\sin(gB\mu_0 t/\hbar) + i\cos(gB\mu_0 t/\hbar) - i\sin(gB\mu_0 t/\hbar)\right)$$

$$\left\langle S_y \right\rangle = \frac{\hbar}{2}\sin(gB\mu_0 t/\hbar)$$

Lastly, for $\left\langle S_z \right\rangle$:

$$\left\langle S_z \right\rangle = \frac{\hbar}{4}\begin{pmatrix} e^{+} & e^{-} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-} \\ e^{+} \end{pmatrix} = \frac{\hbar}{4}\left[e^{+}e^{-} - e^{-}e^{+}\right]$$

$$\left\langle S_z \right\rangle = 0$$

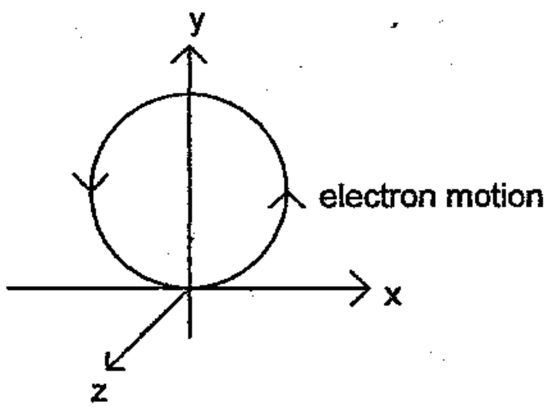
Thus, the expectation of the electron's spin is:

$$\left\langle \vec{S} \right\rangle = \frac{\hbar}{2}\cos\left(gB\mu_0 t/\hbar\right)\hat{x} - \frac{\hbar}{2}\sin\left(gB\mu_0 t/\hbar\right)\hat{y}$$

The electron undergoes cyclotron motion in the magnetic field, and its motion can be calculated classically. The time dependence of the momentum is given by:

$$\vec{p} = p\left[\cos\left(\frac{qBt}{m}\right)\hat{x} + \sin\left(\frac{qBt}{m}\right)\hat{y}\right]$$

Where qB/m is the cyclotron frequency



To find the angle between $\vec{p}$ and $\left\langle \vec{S} \right\rangle$ use $\vec{p}\cdot\left\langle \vec{S} \right\rangle = \left|\vec{p}\right|\left|\left\langle \vec{S} \right\rangle\right|\cos\theta$

Where:

$$\left|\vec{p}\right| = p\sqrt{\cos^2\left(\frac{qBt}{m}\right) + \sin^2\left(\frac{qBt}{m}\right)} = p$$

$$\left|\left\langle \vec{S} \right\rangle\right| = \frac{\hbar}{2}\sqrt{\cos^2\left(\frac{gB\mu_0 t}{\hbar}\right) + \sin^2\left(\frac{gB\mu_0 t}{\hbar}\right)} = \frac{\hbar}{2}$$

Thus:

$$\vec{p}\cdot\left\langle \vec{S} \right\rangle = \frac{\hbar p}{2}\left[\cos\left(\frac{qBt}{m}\right)\cos\left(\frac{gB\mu_0 t}{\hbar}\right) + \sin\left(\frac{qBt}{m}\right)\sin\left(\frac{gB\mu_0 t}{\hbar}\right)\right]$$

$$\vec{p}\cdot\left\langle \vec{S} \right\rangle = \frac{\hbar p}{2}\cos\left[\left(\frac{qB}{m} - \frac{gB\mu\ t}{\hbar}\right)t\right] = \frac{\hbar p}{2}\cos\theta$$

Comparison yields:

$$\theta(t) = \left(\frac{qB}{m} - \frac{gB\mu_0}{\hbar}\right)t$$