Fall 98

(1) e^- in reigon of $\vec{B} = B\hat{z}$. At $t=0$, $\vec{p} \parallel \vec{s} \parallel \hat{x}$.

Let θ be the angle between momentum and $\langle S \rangle$, so at $t=0 \Rightarrow \theta=0$. gyromagnetic ratio is g. What is $\theta(t)$? $p(t)$ can be computed classically

For the spin: $H = -\vec{\mu}\cdot\vec{B} = \frac{g\mu_0}{\hbar}\vec{S}\cdot\vec{B} = g\mu_0 \frac{\sigma_z}{2} B$

initially spin is in the $+\hat{x}$ direction: $\sigma_x |\alpha\rangle = +|\alpha\rangle$

$$0 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \Rightarrow \begin{matrix} \alpha_2 - \alpha_1 = 0 \\ \alpha_1 - \alpha_2 = 0 \end{matrix} \Rightarrow \alpha_1 = \alpha_2$$

normalize $\Rightarrow |\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix}$; $1 = |\langle \alpha|\alpha\rangle|^2 = 2\alpha_1^2 \Rightarrow \alpha_1 = \frac{1}{\sqrt{2}}$.

so $\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (which we could've guessed)

then time evolve: $|\psi(t)\rangle = U|\alpha\rangle = e^{-\frac{iHt}{\hbar}} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-\frac{it}{\hbar} g\mu_0 \frac{\sigma_z}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and knowing $\sigma_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\sigma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(-Bitg\mu_0/2\hbar) \\ \exp(+Bitg\mu_0/2\hbar) \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} e^- \\ e^+ \end{pmatrix}$$

then, $\langle \vec{S} \rangle = \langle S_x \rangle \hat{x} + \langle S_y \rangle \hat{y} + \langle S_z \rangle \hat{z}$

so that $\langle S_x \rangle = \frac{\hbar}{4} \begin{pmatrix} e^+ & e^- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^- \\ e^+ \end{pmatrix}$

$$= \frac{\hbar}{4}\left((e^+)^2 + (e^-)^2\right)$$

and $\langle S_y \rangle = \frac{\hbar}{4} \begin{pmatrix} e^+ & e^- \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^- \\ e^+ \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} e^+ & e^- \end{pmatrix} \begin{pmatrix} -ie^+ \\ ie^- \end{pmatrix} = \frac{\hbar}{4}\left(-i(e^+)^2 + i(e^-)^2\right)$

$$\langle S_z \rangle = \frac{\hbar}{4} \begin{pmatrix} e^+ & e^- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^- \\ e^+ \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} e^+ & e^- \end{pmatrix} \begin{pmatrix} e^- \\ -e^+ \end{pmatrix} = \frac{\hbar}{4}\left(e^+e^- - e^-e^+\right) = 0$$

so now:

$$\vec{p} = p\left(\cos\left(\frac{qBt}{m}\right)\hat{x} + \sin\left(\frac{qBt}{m}\right)\hat{y}\right)$$

$$(e^+)^2 = e^{-Bitg\mu_0/\hbar} = \cos(Btg\mu_0/\hbar) - i\sin(Btg\mu_0/\hbar)$$

$$(e^-)^2 = \cos(Btg\mu_0/\hbar) + i\sin(Btg\mu_0/\hbar)$$

$$\theta = \left(\frac{qB}{m} - \frac{qB\mu}{\hbar}\right)t$$

so $\langle S_x \rangle = \frac{\hbar}{2}\cos(Btg\mu_0/\hbar)$

and $\langle S_y \rangle = -\frac{\hbar}{2}\sin(Btg\mu_0/\hbar)$

so $\langle \vec{S} \rangle = \frac{\hbar}{2}\left(\hat{x}\cos(\omega' t) - \hat{y}\sin(\omega' t)\right)$ $\qquad \omega' = \frac{Bg\mu_0}{\hbar}$

now, the electron will precess in the xy plane in a circle, starting at $t=0$ in the x-direction:

$$\vec{p} = p\left(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right) \qquad (+\hat{y} \text{ because } B \sim +\hat{z})$$

can find ω with $q\vec{v}\times\vec{B} = qvB = m\frac{v^2}{R}$ but $v = \omega R$

$$q\omega \cancel{R} B = m\frac{\omega^2 \cancel{R^2}}{\cancel{R}} \Rightarrow \omega = \frac{qB}{m}$$

then, use $\vec{p}\cdot\langle\vec{S}\rangle = |p||S|\cos\theta$

$$p\cos(\omega t)\frac{\hbar}{2}\cos(\omega' t) - p\sin(\omega t)\frac{\hbar}{2}\sin(\omega' t) = \left(\underbrace{\sqrt{(p\cos(\omega t))^2 + (p\sin(\omega t))^2}}_{p}\right)\cdot$$

$$\left(\underbrace{\sqrt{\left(\frac{\hbar}{2}\cos(\omega' t)\right)^2 + \left(\frac{\hbar}{2}\sin(\omega' t)\right)^2}}_{\frac{\hbar}{2}}\right)\cos\theta$$

$$\Rightarrow \frac{\cancel{p}\cancel{\hbar}}{\cancel{2}}\cos(\omega t + \omega' t) = \frac{\cancel{p\hbar}}{\cancel{2}}\cos\theta$$

$$\Rightarrow \boxed{\theta = (\omega + \omega')t = \left(\frac{q}{m} + \frac{q\mu_0}{\hbar}\right)Bt = \left(\frac{g\mu_0}{\hbar} - \frac{e}{m}\right)Bt}$$