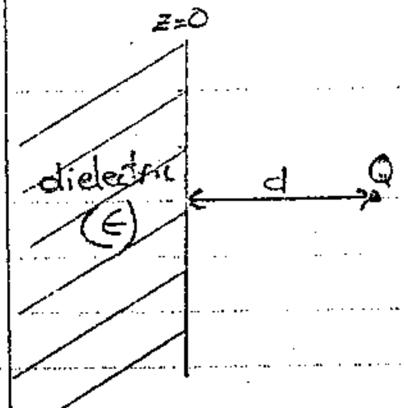


A charge Q sits a distance d outside of a dielectric having dielectric constant ε. What is the electrostatic energy of this configuration?

(refer to Greiner pgs 138-140)

for the two regions we must find solutions to:

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad z>0$$
$$\epsilon(\nabla\cdot\vec{E}) = 0 \qquad z<0$$

and $\nabla\times\vec{E} = 0$ everywhere

with the boundary conditions: $(\vec{D}_2-\vec{D}_1)\cdot\hat{n}_{21} = \sigma_{free}$

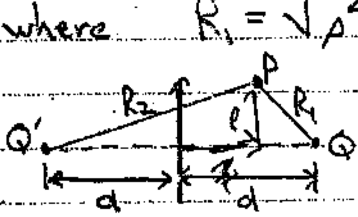
$$(\vec{E}_2-\vec{E}_1)\times\hat{n}_{21} = 0$$

one can write $\lim_{z\to0^+}\begin{Bmatrix}E_z\\E_\rho\end{Bmatrix} = \lim_{z\to0^-}\begin{Bmatrix}\epsilon E_z\\E_\rho\end{Bmatrix}$ ($D_\perp$ continuous for σ_f; $E_\parallel$ continuous)

One can now use image charges to find the potential for $z>0$

$$\Phi = \left(\frac{Q}{R_1}+\frac{Q'}{R_2}\right) \qquad z>0$$

where $R_1 = \sqrt{\rho^2+(d-z)^2}$, $R_2 = \sqrt{\rho^2+(d+z)^2}$



for $z<0$, the potential can be found by placing image cha at location of Q, so

$$\Phi = \frac{Q''}{\epsilon R_1} \qquad z<0$$

The gradient in cylindrical coordinates is:

$$\nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} + \frac{\partial\Phi}{\partial z}\hat{z}$$

so for $z>0$

$$\vec{E} = -\nabla\Phi = -\frac{\partial}{\partial\rho}\left(\frac{Q}{\sqrt{\rho^2+(d-z)^2}}+\frac{Q'}{\sqrt{\rho^2+(d+z)^2}}\right)\hat{\rho} - \frac{\partial}{\partial z}\left(\frac{Q}{\sqrt{\rho^2+(d-z)^2}}+\frac{}{\sqrt{}}\right.$$

$$\lim_{z\to 0^+} \vec{E}(z>0) = \left(\frac{\rho(Q+Q')}{(\rho^2+d^2)^{3/2}}\right)\hat{\rho} + \left(\frac{d(Q'-Q)}{(\rho^2+d^2)^{3/2}}\right)\hat{z}$$

for $z<0$

$$\lim_{z\to 0^-} \vec{E}(z<0) = \left(\frac{\rho Q''}{\epsilon(\rho^2+d^2)^{3/2}}\right)\hat{\rho} + \left(\frac{-d Q''}{\epsilon(\rho^2+d^2)^{3/2}}\right)\hat{z}$$

using the boundary conditions yields:

$$Q + Q' = \frac{Q''}{\epsilon} \quad ; \quad Q - Q' = Q''$$

Solving for the magnitude of the image charges yields

$$Q + Q' = (Q - Q')/\epsilon$$

$$Q(\epsilon-1) + Q'(\epsilon+1)$$

$$\boxed{Q' = \frac{Q(1-\epsilon)}{(1+\epsilon)}}$$

thus the force on the charge in $z>0$ is:

$$\boxed{F = \frac{Q^2(1-\epsilon)}{4d^2(1+\epsilon)}}$$

the sign of the force is nega

The energy is equal to the total work done to bring the charge from $+\infty$ to d.

$$U = \int_{\infty}^{d} \frac{Q^2(1-\epsilon)}{4z^2(1+\epsilon)}\,dz$$

$$U = \frac{Q^2(1-\epsilon)}{4(1+\epsilon)}\left[-z^{-1}\right]_{\infty}^{d} = \frac{-Q^2(1-\epsilon)}{4(1+\epsilon)}\left[\frac{1}{\infty} - \frac{1}{d}\right]$$

$$\boxed{U = \frac{Q^2(1-\epsilon)}{4d(1+\epsilon)}}$$

the energy is negative since the force is attractive.