Spring 1998 #10

Consider a free electron in a static uniform magnetic field given by: $\vec{B} = (B\sin\theta,\ 0,\ B\cos\theta)$
Neglect the energy due to translational motion. At $t=0$, a measurement of the z component of the electron's spin is made, and the value $+\hbar/2$ is obtained. Calculate the probability $P(t)$ that a measurement of the z component of the electron's spin at time $t \geq 0$ will yield $+\hbar/2$. Examine result for $\theta = 0, \frac{\pi}{2}$

$$H = -\vec{\mu}\cdot\vec{B} = \frac{g\mu_o}{\hbar}\vec{B}\cdot\vec{S} = \frac{g\mu_o}{2}\vec{B}\cdot\vec{\sigma}$$

for $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
the Hamiltonian becomes
$$H = \frac{g\mu_o}{2}\begin{pmatrix} B\cos\theta & B\sin\theta \\ B\sin\theta & -B\cos\theta \end{pmatrix}$$
the initial state is found from:
$$\frac{\hbar}{2}(\sigma_z - I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} a = \text{constant} \\ b = 0 \end{matrix}$$

so $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\therefore$ $|\psi(t)\rangle$ is given by: $\quad |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$

$$|\psi(t)\rangle = \exp\left\{\frac{-ig\mu_o Bt}{2\hbar}\left(\sin\theta\,\sigma_x + \cos\theta\,\sigma_z\right)\right\}|+z\rangle$$

we need to express $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of the eigenstates of σ before continuing.
for σ_x;
$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda = \pm 1$$
$\lambda = +1$ yields
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$\lambda = -1$ yields
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|+x\rangle + |-x\rangle\right)$

... for $|\psi(t)\rangle$

$$|\Psi(t)\rangle = \left[\exp\left\{\frac{-iq\mu_0 Bt}{2\hbar}\cos\Theta\ \sigma_z\right\}|+z\rangle\right]\left[\exp\left\{\frac{-iq\mu_0 Bt}{2\hbar}\sin\Theta\ \sigma_x\right\}\frac{(|+x\rangle+|-x\rangle}{\sqrt{2}}$$

$$|\Psi(t)\rangle = \exp\left(\frac{-iq\mu_0 Bt}{2\hbar}\cos\Theta\right)\left[\frac{\exp}{\sqrt{2}}\left(\frac{-iq\mu_0 Bt}{2\hbar}\sin\Theta\right)|+x\rangle + \frac{\exp}{\sqrt{2}}\left(\frac{+iq\mu_0 Bt}{2\hbar}\sin\Theta\right)|-x$$

Now $P_{\hbar/2}(t) = |\langle z|\Psi(t)\rangle|^2$

$$\langle z|\Psi(t)\rangle = e^{-i\alpha\cos\Theta}\left[\left(\frac{e^{-i\alpha\sin\Theta}}{\sqrt{2}}\right)(1\ \ 0)\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} + \left(\frac{e^{+i\alpha\sin\Theta}}{\sqrt{2}}\right)(1\ \ 0)\begin{pmatrix}1/\\ -1/\end{pmatrix}\right.$$

where $\alpha = \frac{iq\mu_0 Bt}{2\hbar}$

$$\langle z|\Psi(t)\rangle = \frac{1}{2}e^{-i\alpha\cos\Theta}\left(e^{-i\alpha\sin\Theta} + e^{i\alpha\sin\Theta}\right)$$

$$|\langle z|\Psi(t)\rangle|^2 = \frac{1}{4}\left[e^{i\alpha\cos\Theta}\left(e^{+i\alpha\sin\Theta} + e^{-i\alpha\sin\Theta}\right)\right]\left[e^{-i\alpha\cos\Theta}\left(e^{-i\alpha\sin\Theta} + e^{i\alpha\sin\right.\right.$$

$$= \frac{1}{4}\left(1 + e^{2i\alpha\sin\Theta} + e^{-2i\alpha\sin\Theta} + 1\right)$$

$$= \frac{1}{4}\left(2 + \cos(2\alpha\sin\Theta) + i\sin(2\alpha\sin\Theta) + \cos(2\alpha\sin\Theta) - i\sin(2\alpha\sin\right.$$

$$\boxed{P_{\hbar/2}(t) = \frac{1}{2}\left[1 + \cos\left(\frac{q\mu_0 Bt}{\hbar}\sin\Theta\right)\right]}$$

when $\Theta = \pi/2$ $\qquad P(t) = \frac{1}{2}\left(1 + \cos\left(\frac{q\mu_0 Bt}{\hbar}\right)\right)$

when $\Theta = 0$ $\qquad P(t) = 1$ $\quad \leftarrow$ stationary state