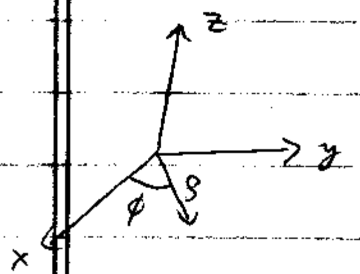
a) An electron of charge e is confined to a circle of radius R. The plane of the ring lies in the x-y plane, perpendicular to the z-axis, with the origin at the center of the ring. Write the classical Hamiltonian for the electron for this problem in cylindrical coordinates (z, ϕ and $\rho = \sqrt{x^2+y^2}$) and their canonical momenta. Then treat this expression as a quantum mechanical Hamiltonian. What are the allowed energy levels?



$$x = \rho\cos\phi \; ; \; y = \rho \sin\phi \; ; \; z = z$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi \Rightarrow \dot{x}^2 = \dot{\rho}^2\cos^2\phi - 2\dot{\phi}\dot{\rho}\rho\sin\phi\cos\phi + \rho^2\dot{\phi}^2\sin^2\phi$

$\dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi \Rightarrow \dot{y}^2 = \dot{\rho}^2\sin^2\phi + 2\dot{\phi}\dot{\rho}\rho\sin\phi\cos\phi + \rho^2\dot{\phi}^2\cos^2\phi$

$\dot{z} = \dot{z} \Rightarrow \dot{z}^2 = \dot{z}^2$

So
$$L = \frac{1}{2} m \Big[\dot{\rho}^2\cos^2\phi \cancel{- 2\dot{\phi}\dot{\rho}\rho\sin\phi\cos\phi} + \rho^2\dot{\phi}^2\sin^2\phi + \dot{\rho}^2\sin^2\phi \cancel{+ 2\dot{\phi}\dot{\rho}\rho\sin\phi\cos\phi} + \rho^2\dot{\phi}^2\cos^2\phi + \dot{z}^2 \Big]$$

$$= \frac{1}{2} m \left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2 \right]$$

The canonical momenta are

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \; ; \; p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \; ; \; p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

The classical Hamiltonian is gotten from:

$$H = \dot\rho p_\rho + \dot\phi p_\phi + \dot z p_z - L$$

$$= \dot\rho \overset{m\dot\rho}{\overset{\Downarrow}{p_\rho}} + \dot\phi \overset{\rho^2 m\dot\phi}{\overset{\Downarrow}{p_\phi}} + \dot z \overset{m\dot z}{\overset{\Downarrow}{p_z}} - \tfrac{1}{2}m\dot\rho^2 - \tfrac{1}{2}m\rho^2\dot\phi^2 - \tfrac{1}{2}m\dot z^2$$

$$= \tfrac{1}{2}m\dot\rho^2 + \tfrac{1}{2}m\rho^2\dot\phi^2 + \tfrac{1}{2}m\dot z^2 = \tfrac{1}{2}m\left[\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\right]$$

now $\dot\rho = 0$ as the particle is moving at a fixed radius R, so (also $\dot z = 0$)

$$H = \tfrac{1}{2}m\left[\overset{R^2}{\overset{\Downarrow}{\rho^2}}\dot\phi^2 + \cancel{\dot z^2}^{\to 0}\right] = \tfrac{1}{2}mR^2\dot\phi^2 + \tfrac{1}{2}m\cancel{\dot z^2}^{\to 0}$$

now $p_\phi = \rho^2 m\dot\phi \Rightarrow p_\phi^2 = \rho^4 m^2\dot\phi^2 \Rightarrow \dot\phi^2 = \dfrac{p_\phi^2}{\underset{R^4}{\rho^4} m^2}$

~~also~~ $\cancel{p_z = m\dot z \Rightarrow p_z^2 = m^2\dot z^2 \Rightarrow \dot z^2 = \frac{p_z^2}{m^2}}$

so

$$H = \tfrac{1}{2}\cancel{m}\cancel{R^2}\frac{p_\phi^2}{R^{\cancel{4}\,2}m^{\cancel{2}}} + \cancel{\tfrac{1}{2}m\frac{p_z^2}{m^2}} = \frac{1}{2mR^2}p_\phi^2 + \cancel{\frac{1}{2m}p_z^2}$$

now $p_\phi = L_z$ so

$$H = \cancel{\frac{p_z^2}{2m}} + \cancel{\frac{1}{2m}\frac{L_z^2}{R^2}} \quad \frac{1}{2m}\frac{L_z^2}{R^2}$$

so the energy levels are:

$$E = \cancel{\frac{\hbar^2 k_z^2}{2m}} + \frac{\hbar^2 n^2}{2mR^2}$$

b) Now let there be an external magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$, constant in time and space. Show that $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$. Take $\vec{B}$ in the z-direction: $\vec{B} = B_0\hat{n}_z$. The Hamiltonian is

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2.$$

What are the quantum energy levels now?

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{2}\vec{\nabla}\times(\vec{B}\times\vec{r}) \qquad \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

or

$$\vec{B}(\vec{A}\cdot\vec{C}) - \cancel{(\vec{B}\cdot\vec{A})}\vec{C}$$

$$= \frac{1}{2}\left[\vec{B}(\vec{\nabla}\cdot\vec{r}) - (\vec{B}\cdot\vec{\nabla})\vec{r}\right]$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\ ; \quad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}\ ; \quad \vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

- $\vec{\nabla}\cdot\vec{r} = 1 + 1 + 1 = 3$
- $(\vec{B}\cdot\vec{\nabla})\vec{r} = \overbrace{\left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)}^{A}(x\hat{x} + y\hat{y} + z\hat{z}) = Ax\hat{x} + Ay\hat{y} + Az\hat{z}$

$$= B_x\hat{x} + B_y\hat{y} + B_z\hat{y} = \vec{B}$$

so $\vec{B} = \frac{1}{2}\left[3\vec{B} - \vec{B}\right] = \vec{B}$ ✓

Now let $\vec{B} = B_0\hat{z}$ so

$$\hat{x}\times\hat{y} = \hat{z}$$

$$\vec{A} = \frac{1}{2}\left(B_0\hat{z}\times[x\hat{x} + y\hat{y} + z\hat{z}]\right) = \frac{1}{2}B_0(x\hat{y} - y\hat{x} + 0)$$

$$= \frac{1}{2}B_0(-y\hat{x} + x\hat{y}) = \left(-\frac{B_0 y}{2}, \frac{B_0 x}{2}, 0\right)$$

expand the Hamiltonian:

$$H\psi = \frac{1}{2m}\left[-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right]^2\psi = \frac{1}{2m}\left[-\hbar^2\nabla^2\psi - i\frac{\hbar e}{c}\vec{\nabla}\cdot(A\psi) - i\frac{\hbar e}{c}\vec{A}\cdot(\vec{\nabla\psi}) + \frac{e^2}{c^2}\vec{A}^2\psi\right]$$

- $\vec{\nabla}\cdot(\vec{A}\psi) = \psi\underbrace{(\vec{\nabla}\cdot\vec{A})}_{=0} + \vec{A}\cdot(\vec{\nabla\psi}) = \vec{A}\cdot(\vec{\nabla\psi})$

- $\vec{A}^2 = \frac{B_0^2}{4}(y^2+x^2)$

so $$H\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{i\hbar}{mc}\vec{A}\cdot(\vec{\nabla\psi}) + \frac{1}{2}\frac{e^2B_0^2}{4mc^2}\psi$$

$$= -\frac{\hbar^2}{2m}\nabla^2\psi - \underbrace{\frac{i\hbar}{mc}\frac{eB_0}{2}\left[-y\frac{\partial}{\partial x}\psi + x\frac{\partial}{\partial y}\psi\right]} + \frac{1}{2}\frac{e^2B_0^2}{4mc^2}\psi(x^2+y^2)$$

$$\frac{eB_0}{2mc}\underbrace{\left[-yp_x + xp_y\right]}_{L_z}\psi$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}\frac{eB_0}{mc}L_z + \frac{1}{2}\frac{e^2B_0^2}{4mc^2}x^2 + \frac{1}{2}\frac{e^2B_0^2}{4mc^2}y^2\right]\psi$$

define $\omega_0 = \frac{eB_0}{mc}$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\frac{\omega_0^2}{4}x^2 + \frac{1}{2}m\frac{\omega_0^2}{4}y^2 + \frac{1}{2}\omega_0 L_z\right]\psi$$

try $\psi = e^{ik_z z}\varphi(x)\varphi(y)Y_\ell^m$

$\hat{z}$: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi = E_z\psi \Rightarrow E_z = \frac{\hbar^2k_z^2}{2m}$

$\hat{x}$: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}m\left(\frac{\omega_0}{2}\right)^2x^2\psi = E_x\psi \Rightarrow E_x = \hbar\frac{\omega_0}{2}(n_x+1/2)$

$\hat{y}$: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m\left(\frac{\omega_0}{2}\right)^2y^2\psi = E_y\psi \Rightarrow E_y = \hbar\frac{\omega_0}{2}(n_y+1/2)$

$\hat{L}_z$: $\frac{1}{2}\omega_0 L_z\psi = \frac{\hbar\omega_0}{2}m\psi \Rightarrow E_{L_z} = \frac{\hbar\omega_0}{2}m$

so $$E = \frac{\hbar^2k_z^2}{2m} + \frac{\hbar\omega_0}{2}\left[(n_x+1/2) + (n_y+1/2) + m\right]$$

if $B_0 = 0$ then $\omega_0 = 0$ $\Rightarrow E = \frac{\hbar^2k_z^2}{2m}$ now $k_z = \frac{2\pi}{\lambda}$; $\lambda = \frac{2\pi R}{n} \Rightarrow k_z = \frac{n}{R} \Rightarrow k_z^2 = \frac{n^2}{R^2}$ so $E = \frac{\hbar^2}{2mR^2}n^2$ as before