Abers #43 (similar to problem 6.1)

(a) write the classical Hamiltonian for a free electron in terms of spherical coordinates and their canonical momenta. Suppose the electron is confined to a circular ring of radius R. The ring lies in the x-y plane, and the z-axis passes through the center of the ring. Impose this constraint on P_r and P_θ. what is the Hamiltonian with this constraint? Then treat this expression as a quantum mechanical Hamiltonian, what are the allowed energy levels?

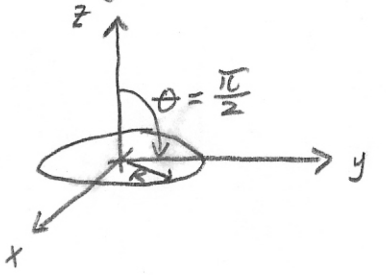
(i) free electron classically is given by

$$H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{P_\phi^2}{2mr^2\sin^2\theta}$$

where the canonical momenta are given by

$P_r = m\dot{r}$, $P_\theta = mr^2\dot{\theta}$, and $P_\phi = m^2\sin^2\theta\ \phi'$ where "$'$" $\equiv \frac{\partial}{\partial t}$

(ii) electron confined to a circular ring of radius R



since $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$
& $r = R$, $\dot{r} = 0$

since $\dot{r} = 0 = \dot{\theta} \Rightarrow P_r = 0 = P_\theta$

note: $P_\phi = mR^2\phi' = L_z$

Thus, the Hamiltonian is given by

$$H = \frac{P_\phi^2}{2mR^2} = \frac{L_z^2}{2mR^2}$$

note; the potential term is "built-in" to the constraints.

(iii) allowed energy levels

For this case, Schrödinger equation is given by (recall $\hbar = 1$)

$$-\frac{1}{2mR^2}\frac{d^2}{d\phi^2}\psi(\phi) - E\,\psi(\phi) = 0 \Rightarrow \frac{d^2}{d\phi^2}\psi(\phi) + n^2\,\psi(\phi) = 0 \ , \ n^2 = 2mR^2E$$

the solutions to this D.E. are

$$\psi(\phi) \propto e^{\pm in\phi}$$

Now, since those functions repeat every 2π, we know that n is an integer.
The energy levels come from the definition of n. That is,

$$n^2 = 2mR^2E$$

$$\therefore \boxed{E_n = \frac{n^2}{2mR^2}} \qquad n = 1, 2, 3, \ldots$$

(b) Now let there be a constant external magnetic field $\vec{B} = \nabla \times \vec{A}$. Take it to be in the z-direction, perpendicular to the plane of the ring: $\vec{B} = B_0 \hat{n}_z$. The Hamiltonian is

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2$$

The components of $\vec{p}$ are still constrained as in part (a). In the transverse gauge

$$\vec{A}(\vec{r}) = \frac{1}{2}\vec{B} \times \vec{r}$$

is the correct form of the vector potential. What are the cylindrical components A_ρ, A_z, and A_ϕ?

(i) $$\vec{B} \times \vec{r} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ 0 & 0 & B_0 \\ \rho & 0 & z \end{vmatrix} = \frac{1}{\rho}\left[\hat{\rho}(0) - \rho\hat{\phi}(-B_0\rho) + \hat{z}(0)\right] = B_0\rho\hat{\phi}\Big|_{\rho=R}$$

where we used $\vec{r} = \rho\hat{\rho} + z\hat{z}$ in cylindrical coordinates. So, we have

$$\vec{A}(\vec{r}) = \frac{1}{2}(\vec{B} \times \vec{r}) = \frac{1}{2}B_0 R\hat{\phi} \qquad (1)$$

Thus,

$$\boxed{A_\rho = 0,\ A_z = 0,\ \text{\&}\ A_\phi = \frac{1}{2}B_0 R}$$

(ii) Hamiltonian in terms of L_z and A_ϕ

note: $$-i\frac{\partial}{\partial\phi} = P_\phi = L_z \quad (2) \qquad \text{and} \quad \vec{p} = \frac{P_\phi}{R}\hat{\phi} = \frac{L_z}{R}\hat{\phi}$$

So,

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 \underset{\substack{\uparrow \\ \text{using} \\ \text{eq (1) \& (2)}}}{=} \frac{1}{2m}\left(\frac{L_z}{R} + \frac{e}{2c}B_0R\right)^2 = \frac{1}{2mR^2}\left(L_z + \frac{e}{2c}B_0R^2\right)^2$$

So, the Schrödinger eq is

$$\frac{1}{2mR^2}\left(-i\frac{\partial}{\partial\phi}+\frac{e}{2c}B_0R^2\right)^2\psi(\phi)=E\,\psi(\phi)$$

So, the energies are (let $c=1$)

$$\boxed{\frac{1}{2mR^2}\left(n+\frac{e}{2}B_0R^2\right)^2=E_n}$$

note: when $B_0=0$, we get the same answer as in part (a)!