Spring 1998 # 12

A particle in 3-D is subject to a potential $V(\vec{r})$ which is non-zero only in a spherical shell of radius a about the origin:

$$V(\vec{r}) = -V_o a\,\delta(r-a)$$

where V_o and a are positive numbers, and δ is the Dirac delta function.

→ The 3-Dimensional radial equation is:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{L^2}{r^2}\right) + V(r)\right]R(r) = E\,R(r)$$

$$\left[-\frac{\hbar^2}{2mr^2}\left(\frac{d}{dr}r^2\frac{d}{dr} - \ell(\ell+1)\right) + V(r)\right]R(r) = E\,R(r)$$

Consider $\ell=0$ case for simplicity

$$\frac{1}{r}\left(\frac{d}{dr}r^2\frac{d}{dr}R(r)\right) + \frac{2m}{\hbar^2}V_o a\,\delta(r-a)R(r) = -\frac{2mE}{\hbar^2}R(r)$$

let $u(r) = r\,R(r) \Rightarrow R(r) = \frac{u(r)}{r}$

$$\frac{1}{r^2}\frac{d}{dr}r^2\left[\frac{u'}{r} - \frac{u}{r^2}\right] + \frac{2m}{\hbar^2}V_o a\,\delta(r-a)\frac{u}{r} = -\frac{2mE}{\hbar^2}\frac{u}{r}$$

$$\frac{1}{r^2}\left[ru'' + \cancel{u'} - \cancel{u'}\right] + \frac{2m}{\hbar^2}V_o a\,\delta(r-a)\frac{u}{r} + \frac{2mE}{\hbar^2}\frac{u}{r} = 0$$

$$u'' + \frac{2m}{\hbar^2}V_o a\,\delta(r-a)\,u + \frac{2mE}{\hbar^2}u = 0$$

for $r<a$:

$$u'' = -\frac{2mE}{\hbar^2}u = k^2 u \qquad k = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$u(r) = A\sinh(kr)$$ ($u(0)$ must be zero Aben pg 7)

for $r>a$:

$$u(r) = B e^{-kr}$$

⇒

Now, continuity of Ψ at $r=a$ requires:

(i) $$B e^{-ka} = A \sinh(ka)$$

at the discontinuity:

$$\lim_{\epsilon \to 0} \int_{a-\epsilon}^{a+\epsilon} u'' \, dr + \int_{a-\epsilon}^{a+\epsilon} \frac{2m}{\hbar^2} V_0 a \, \delta(r-a) \, dr = -\int_{a-\epsilon}^{a+\epsilon} \frac{2mE}{\hbar^2} U \, dr$$

$$u'\Big]_{a^+} - u'\Big]_{a^-} + \frac{2m}{\hbar^2} V_0 a \, U(a) = 0$$

(ii) $$-kBe^{-ka} - kA\cosh(ka) = -\frac{2m}{\hbar^2} V_0 a \, B e^{-ka}$$

substituting (i) into (ii) yields

$$-k\cancel{A}\sinh(ka) - k\cancel{A}\cosh(ka) = -\frac{2m}{\hbar^2} V_0 a \, \cancel{A}\sinh(ka)$$

$$\sinh(ka)\left[\frac{2m}{\hbar^2} V_0 a - k\right] = +k\cosh(ka)$$

$$\tanh(ka) = \left(\frac{2m}{\hbar^2} V_0 a - k\right)^{-1} k$$

since $ka > 0$, $0 < \tanh(ka) < 1$

so for a bound state to occur:

$$0 < k\left(\frac{2m}{\hbar^2} V_0 a - k\right)^{-1} < 1$$

$$0 < \frac{\hbar^2 k}{2mV_0 a - k\hbar^2} < 1$$

$$0 < \hbar^2 k < 2mV_0 a - k\hbar^2$$

$$\hbar^2 k < 2\hbar^2 k < 2mV_0 a \quad \Rightarrow \quad V_0 > \frac{\hbar^2 k}{2ma}$$ use $k=\sqrt{}$

$$V_0^2 > \frac{\hbar^4}{4m^2a^2}\left(\frac{-2mE}{\hbar^2}\right)$$ for minimum V_0, $E = -V_0$

$$V_0^2 > \frac{\hbar^2 V_0}{2ma^2} \quad \Rightarrow \quad \boxed{V_0 > \frac{\hbar^2}{2ma^2}}$$

thus, the existence of a bound state depends on the values of V_0 and a