# Solution to Problem #12 from Spring 1998 Comps

12) (Quantum Mechanics) A particle in 3-D is subject to a potential $V(\vec{r})$ which is non-zero only in a spherical shell of radius $a$ about the origin:

$$V(\vec{r}) = -V_0\, a\, \delta(r-a),$$

where $V_0$ and $a$ are positive numbers, and $\delta$ is the Dirac delta-function. Show which of the following statements are true and which are false:

a) There are no values of $V_0$ and $a$ for which there is a bound state.

b) There is a bound state no matter what the values of $V_0$ and $a$ are.

c) The existence of a bound state depends on the values of $V_0$ and $a$. (If you say this is true, then give the condition on $V_0$ and $a$ in order for a bound state to exist.)

The radial wave equation is

$$\left[\frac{1}{r^2}\frac{d}{dr} r \frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right] R_l(r) = K^2 R_l(r)$$

where $K^2 = -2mE/\hbar^2$ and $V(r) = -V_0 a\, \delta(r-a)$

to simplify define

$$\lambda = 2mV_0 a^2/\hbar^2 \quad \text{and} \quad u_l(r) = r R_l(r)$$

Then we look at the $l=0$ case.

$$\frac{d^2}{dr^2} u(r) + \left[\frac{\lambda}{a}\delta(r-a) - K^2\right] u(r) = 0$$

Now we find the solutions for $r \neq a$

$$\frac{d^2}{dr^2} u(r) = K^2 u(r)$$

solution

$$u(r) = A' e^{Kr} + B' e^{-Kr}$$

$u(r)$ must vanish at the origin and at infinity.

Therefore, the solutions are

for $r < a$

$$u(r) = A \sinh(Kr)$$

for $r > a$

$$u(r) = B e^{-Kr}$$

Matching the solutions across the boundary $r = a$ (due to the continuity condition)

$$A \sinh(Ka) = B e^{-Ka}$$

also

$$\lim_{\epsilon \to 0} \left\{ \int_{a-\epsilon}^{a+\epsilon} u''(r)\, dr + \int_{a+\epsilon}^{a+\epsilon} \left( \frac{\lambda}{a} \delta(r-a)\, u(r) - K^2 u(r) \right) dr \right\} = 0$$

$$u'(r)\Big|_{a^+} - u'(r)\Big|_{a-} + \frac{\lambda}{a} u(a) - 0 = 0$$

$$-KBe^{-Ka} - KA\cosh(Ka) + \frac{\lambda}{a} Be^{-Ka} = 0$$

combine with $A\sinh(Ka) = Be^{-Ka}$

to get $-KA(\sinh(Ka) + \cosh(Ka)) + \frac{\lambda}{a} A\sinh(Ka) = 0$

$$1 + \frac{\cosh(Ka)}{\sinh(Ka)} = \frac{\lambda}{Ka}$$

or

$$\sinh(Ka)\left[\frac{\lambda}{a} - K\right] = K\cosh(Ka)$$

$$\Rightarrow \tanh(Ka) = \frac{K}{\left[\frac{\lambda}{a} - K\right]} = \frac{Ka}{[\lambda - Ka]}$$

This is the condition for a bound state which is a transcendental equation for $Ka$.

you can plot this transcendental equation to find the solution

$$1 > \frac{1}{\lambda}$$

$$V_0 > \frac{\hbar^2}{2ma^2}$$

Or we can note that $\kappa a > 0$

Therefore to get a bound state

$$0 < \tanh(\kappa a) < 1 \Rightarrow \lambda > \kappa a$$

$$\cancel{0 < \frac{\kappa a}{[\lambda - \kappa a]} < 1}$$

$$\cancel{0 < \kappa a < \lambda - \kappa a}$$

$$\cancel{\kappa a < 2\kappa a < \lambda}$$

$$\Rightarrow \boxed{V_0 > \frac{\hbar^2}{2ma^2}}$$

Therefore the answers are

a) ~~Fasle~~ False

b) False

c) True $\boxed{V_0 > \frac{\hbar^2}{2ma^2}}$