

A system of 2 particles, each with possible energies 0, ϵ , 3ϵ is in contact with a reservoir at temperature T .

What is the mean number of particles in the state with energy ϵ for:

a) distinguishable particles (Maxwell-Boltzmann statistics)?

In general, the mean number of particles in a state s is given by:

$$\bar{n}_s = -\frac{1}{\beta} \frac{\partial \ln z}{\partial \epsilon_s} \quad (\text{Reif eq 9.2.5})$$

Where z = partition function

A system of N particles obeying MB statistics has the partition function

$$z = (z_1)^N$$

Where z_1 = partition function of a single particle

For the given case:

$$z_1 = e^{-\beta E_0} + e^{-\beta E_1} + e^{-\beta E_2}$$

Where $E_0 = 0$ $E_1 = \epsilon$ $E_2 = 3\epsilon$

Note that it is important not to express z in terms of ϵ at this point, because doing so would not allow one to take the derivative with respect to a specific energy.

$$z = (z_1)^2 = (e^{-\beta E_0} + e^{-\beta E_1} + e^{-\beta E_2})^2$$

$$\ln z = 2 \ln z_1$$

$$\bar{n}_\epsilon = -\frac{1}{\beta} \frac{\partial \ln z}{\partial E_1} = -\frac{2}{\beta z_1} (-\beta e^{-\beta E_1})$$

Now one can substitute for the values of E to obtain:

$$\bar{n}_\epsilon = \frac{2e^{-\beta \epsilon}}{1 + e^{-\beta \epsilon} + e^{-3\beta \epsilon}}$$

b) particles obeying Bose-Einstein statistics?

The approach for this scenario is the same as in (a), but in order to find z , it is first useful to list the possible states for the system:

$0 = E_0$	$\epsilon = E_1$	$3\epsilon = E_2$	Total E
AA			0
	AA		2ϵ
		AA	6ϵ
A	A		ϵ
A		A	3ϵ
	A	A	4ϵ

There are six unique states for this system. Thus, the partition function is:

$$z = e^{-2\beta E_0} + e^{-2\beta E_1} + e^{-2\beta E_2} + e^{-\beta(E_0 + E_1)} + e^{-\beta(E_0 + E_2)} + e^{-\beta(E_1 + E_2)}$$

$$\bar{n}_\varepsilon = -\frac{1}{\beta} \frac{\partial \ln z}{\partial E_1} = -\frac{1}{\beta z} \left(-2\beta e^{-\beta E_1} - \beta e^{-\beta(E_0+E_1)} - \beta e^{-\beta(E_1+E_2)} \right)$$

$$\bar{n}_\varepsilon = \frac{3e^{-\beta\varepsilon} + e^{-4\beta\varepsilon}}{1 + e^{-2\beta\varepsilon} + e^{-6\beta\varepsilon} + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon}}$$

c) particles obeying Fermi-Dirac statistics?

The allowed states are:

$0=E_0$	$\varepsilon=E_1$	$3\varepsilon=E_2$	Total E
A	A		ε
A		A	3ε
	A	A	4ε

There are only three unique states, so

$$z = e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta(E_1+E_2)}$$

$$\bar{n}_\varepsilon = -\frac{1}{\beta} \frac{\partial \ln z}{\partial E_1} = -\frac{1}{\beta z} \left(-\beta e^{-\beta E_1} - \beta e^{-\beta(E_1+E_2)} \right)$$

$$\bar{n}_\varepsilon = \frac{e^{-\beta\varepsilon} + e^{-4\beta\varepsilon}}{e^{-\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon}}$$

$$\bar{n}_\varepsilon = \frac{1 + e^{-3\beta\varepsilon}}{1 + e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon}}$$