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① We want to solve
$$\nabla\cdot\vec{D} = \varepsilon_0 \nabla\cdot\vec{E} = \rho \;, \quad z>0$$
$$\nabla\cdot\vec{D} = \varepsilon \nabla\cdot\vec{E} = 0 \;, \quad z<0$$
$$\nabla\times\vec{E} = 0$$

and due to the boundary conditions we require
$$\lim_{z\to 0^-}\begin{Bmatrix}\varepsilon E_z \\ E_\rho\end{Bmatrix} = \lim_{z\to 0^+}\begin{Bmatrix}\varepsilon_0 E_z \\ E_\rho\end{Bmatrix} \qquad \text{(clearly no } E_\phi)$$

Since $\nabla\times\vec{E}=0$ write down Φ. Try obvious image charges:

(I) $z>0$: image Q' at $z=-d$. $\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R_1}+\frac{Q'}{R_2}\right)$

with $R_1 = \sqrt{(z-d)^2+\rho^2}$ $\qquad R_2=\sqrt{(z+d)^2+\rho^2}$

then
$$\vec{E} = -\nabla\Phi = -\left(\hat{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\hat{\phi}\cancel{\frac{\partial}{\partial\phi}} + \hat{z}\frac{\partial}{\partial z}\right)\Phi = \frac{-1}{4\pi\varepsilon_0}\left(\hat{\rho}\left(\frac{\partial}{\partial\rho}\frac{Q}{R_1}+\frac{\partial}{\partial\rho}\frac{Q'}{R_2}\right)\right.$$
$$\left. + \left(\hat{z}\left(\frac{\partial}{\partial z}\frac{Q}{R_1}+\frac{\partial}{\partial z}\frac{Q'}{R_2}\right)\right)\right)$$
$$= \frac{+1}{4\pi\varepsilon_0}\left(\hat{\rho}\left(Q\frac{\rho}{((z-d)^2+\rho^2)^{3/2}} + Q'\frac{\rho}{((z+d)^2+\rho^2)^{3/2}}\right) + \hat{z}\left(Q\frac{z-d}{((z-d)^2+\rho^2)^{3/2}} + Q'\frac{z+d}{((z+d)^2+\rho^2)^{3/2}}\right)\right)$$

(II) $z<0$: image Q'' at $z=+d$:
$$\Phi = \frac{1}{4\pi\varepsilon}\frac{Q''}{R_1} \Rightarrow \vec{E} = +\frac{1}{4\pi\varepsilon}\left(\hat{\rho}\,Q''\frac{\rho}{((z-d)^2+\rho^2)^{3/2}} + \hat{z}\,Q''\frac{z-d}{((z-d)^2+\rho^2)^{3/2}}\right)$$

at $z=0$, require $\varepsilon_0 E_z(z>0) = \varepsilon E_z(z<0) \Rightarrow \frac{+\cancel{\varepsilon_0}}{\cancel{4\pi\varepsilon_0}}\left(Q\frac{-d}{(d^2+\rho^2)^{3/2}}+Q'\frac{d}{(d^2+\rho^2)^{3/2}}\right)$
$$= \frac{+\cancel{\varepsilon}}{\cancel{4\pi\varepsilon}}\left(Q''\frac{-d}{(d^2+\rho^2)^{3/2}}\right)$$

$$\therefore \boxed{+Q-Q' = +Q''}$$

also require $E_\rho = E_\rho$: $\frac{+1}{4\pi\varepsilon_0}\left(\frac{Q\cancel{\rho}}{(\cancel{d^2+\rho^2})^{3/2}}+\frac{Q'\cancel{\rho}}{(\cancel{d^2+\rho^2})^{3/2}}\right) = \frac{+1}{4\pi\varepsilon}\left(\frac{Q''\cancel{\rho}}{(\cancel{d^2+\rho^2})^{3/2}}\right) \Rightarrow \varepsilon(Q+Q') = \varepsilon_0 Q''$

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we care about the energy, i.e. the force, i.e. the field on Q, which is in (I), so

$$Q - Q' = Q'' = \frac{\varepsilon}{\varepsilon_0} Q + \frac{\varepsilon}{\varepsilon_0} Q' \Rightarrow Q\left(1 - \frac{\varepsilon}{\varepsilon_0}\right) = Q'\left(\frac{\varepsilon}{\varepsilon_0} + 1\right) \Rightarrow Q' = \frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon} Q$$

could solve for the field here, but force between 2 point charges is just:

$$F = \frac{+1}{4\pi\varepsilon_0} \frac{QQ'}{r^2} = \frac{-Q^2}{16\pi\varepsilon_0 d^2}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right) \qquad \text{(negative because attractive)}$$

and the work to assemble this is just

$$U = -\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)\frac{Q^2}{4\pi\varepsilon_0}\int_\infty^{2d} \frac{1}{r^2}\, dr = -\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)\frac{Q^2}{4\pi\varepsilon_0}\left(-\frac{1}{2d}\right)$$

For fun, the long way: $\rho = 0$ axis component of E:

$$E_z = \frac{+Q}{4\pi\varepsilon_0}\left(\frac{z-d}{(z-d)^3} + \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)\frac{z+d}{(z+d)^3}\right)$$

so that at $z = d$, the force is $F = QE = \frac{-Q^2}{4\pi\varepsilon_0}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)\frac{1}{4d^2} = \frac{-Q^2}{16\pi\varepsilon_0 d^2}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)$

and $U = \frac{-Q^2}{4\pi\varepsilon_0}\int_\infty^d \frac{1}{(z+d)^2}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right) dz = \frac{-Q^2}{4\pi\varepsilon_0}\left(-\frac{1}{z+d}\right)\Big|_\infty^d \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)$ (ignoring self-energy)

or just use $U = Q\Phi = \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)\frac{1}{R_1}\Big|_{\vec{r} = d\hat{z}} = \boxed{\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\right)\frac{1}{2d}}$