Consider a neutral sheet of time-dependent current which is uniform over the entire y-z plane at $x=0$.

$$\vec{J} = J_0 \cos(\omega t)\,\delta(x)\,\hat{y}$$

a) Write down or derive the jump in $\vec{E}$ and $\vec{B}$ across the $x=0$ plane.

→ The boundary conditions at any interface are:

$$\vec{D}_{1\perp} - \vec{D}_{2\perp} = 4\pi\sigma_f \qquad \vec{H}_{1\parallel} - \vec{H}_{2\parallel} = \frac{4\pi}{c}\vec{K}_f \times \hat{n} = \frac{4\pi}{c} J_0 \cos(\omega t)\delta(x)(\hat{y}$$

$$\vec{E}_{1\parallel} - \vec{E}_{2\parallel} = 0 \qquad B_{1\perp} - B_{2\perp} = 0$$

since we are in free space $\vec{D} = \vec{E}$, $\vec{H} = \vec{B}$; also $\sigma_f = 0$, so the boundary conditions become:

$$\boxed{\begin{array}{ll} \vec{E}_{1\perp} - \vec{E}_{2\perp} = 0 & \vec{B}_{1\parallel} - \vec{B}_{2\parallel} = -\frac{4\pi}{c} J_0 \cos(\omega t)\,\hat{z} \\ \vec{E}_{1\parallel} - \vec{E}_{2\parallel} = 0 & \vec{B}_{1\perp} - \vec{B}_{2\perp} = 0 \end{array}}$$

b) Calculate $\vec{E}$ and $\vec{B}$ for $x>0$ and for $x<0$

Maxwell's equations in free space are:

(i) $\nabla \cdot \vec{E} = 0$ (iii) $\nabla \cdot \vec{B} = 0$

(ii) $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ (iv) $\nabla \times \vec{B} = \frac{1}{c}\frac{\partial E}{\partial t}$

taking the curl of (ii), (iv) yields:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \frac{1}{c}\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \frac{1}{c}\nabla \times \left(\frac{\partial E}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{E}) = -\frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

Thus, it is confirmed that $\vec{E}, \vec{B}$ satisfy the wave equation. Since the waves must propagate along the x-axis, the solution will be of the

$$B_{>0}(x,t) = B_+ \cos(kx - \omega t)\,\hat{z}$$

$$E_{>0}(x,t) = E_+ \cos(kx - \omega t)\,\hat{y}$$

$$B_{<0}(x\,t) = B_- \cos(kx - \omega t)(-\hat{z})$$

$$E_{<0}(x,t) = E_- \cos(kx - \omega t)(\hat{y})$$

$E_+ = E_-$

to satisfy $B_{1\parallel} - B_{2\parallel} = \frac{-4\pi}{c} J_0 \cos(\omega t)\,\hat{z}$ requires

$$-B_- \cos(\omega t) - B_+ \cos(\omega t) = -\frac{4\pi}{c} J_0 \cos(\omega t)$$

which yields $B_- + B_+ = \frac{4\pi}{c} J_0$

now recall that $E_+ = B_+$, $E_- = B_-$ from $\vec{B} = \hat{k} \times \vec{E}$

thus $2B_+ = \frac{4\pi}{c} J_0$

so $B_+ = \frac{2\pi}{c} J_0 = B_- = E_+ = E_-$

Finally, one gets (with $k = \frac{\omega}{c}$)

$$\boxed{\begin{aligned} \vec{B}_{x>0}(x,t) &= \tfrac{2\pi}{c} J_0 \cos(\tfrac{\omega}{c}x - \omega t)\,\hat{z} \\ \vec{E}_{x>0}(x,t) &= \tfrac{2\pi}{c} J_0 \cos(\tfrac{\omega}{c}x - \omega t)\,\hat{y} \\ \vec{B}_{x<0}(x,t) &= -\tfrac{2\pi}{c} J_0 \cos(\tfrac{\omega}{c}x - \omega t)\,\hat{z} \\ \vec{E}_{x<0}(x,t) &= \tfrac{2\pi}{c} J_0 \cos(\tfrac{\omega}{c}x - \omega t)\,\hat{y} \end{aligned}}$$

c) Calculate the energy flux for $x > 0$, find the time averaged radiated power per unit area, and the work done by the $\vec{E}$ field on the charge per unit area.

$$\boxed{\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{\cancel{c}}{\cancel{4\pi}}\left(\frac{4\pi^2}{c^{\cancel{2}}}\right) J_0^2 \cos^2(\tfrac{\omega}{c}x - \omega t)} = \text{energy flux}/(\text{time}\cdot\text{Area})$$

$$\langle \vec{S} \rangle = \pi J_0^2 \langle \cos^2(\tfrac{\omega}{c}x - \omega t) \rangle$$

$$\boxed{\langle S \rangle = \frac{\pi}{2} J_0^2}$$

Work done by electric field $= \vec{J}\cdot\vec{E}$

$$\vec{J}\cdot\vec{E} = \left(J_0 \cos(\omega t)\,\delta(x)\,\hat{y}\right)\cdot\left(\tfrac{2\pi}{c} J_0 \cos(\tfrac{\omega}{c}x - \omega t)\,\hat{y}\right)$$

$$= \frac{2\pi}{c} J_0^2 \cos(\omega t) \cos(\tfrac{\omega}{c}x - \omega t)\,\delta(x)$$

Integrate over x to find work done per unit area in y-z plane.

$$\int_{-\infty}^{+\infty} J\cdot E\,dx = \boxed{\frac{2\pi}{c} J_0^2 \cos^2(\omega t) = W}$$