Consider the motion of a charge in an electric field $\vec{E}$, using the correct equations of special relativity. Let $\vec{E} = E_0 \hat{x}$ with E_0 constant. Consider a particle with initial conditions $x = y = z = v_x = v_z = 0$ at $t=0$, but with $v_y \neq 0$ at $t=0$.

a) Solve for $v_x(t)$, $v_y(t)$, and $v_z(t)$. Also find simplified expressions for v_x, v_y when $t \to \infty$. (refer to Greiner pgs 490-491)

→ Start from $\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$ where $\vec{p}$ is the relativistic momentum. For each component this yields:

$$\frac{dp_x}{dt} = qE_0 \qquad \frac{dp_y}{dt} = 0 \qquad \frac{dp_z}{dt} = 0$$

the components of $\vec{p}$ as functions of time are thus

$$p_x = qE_0 t \qquad p_y = p_i \qquad p_z = 0$$

One can now write the momentum 4-vector p_μ:

$$p_\mu = \left(\frac{T}{c}, qE_0 t, p_i, 0\right) = (p_0, p_1, p_2, p_3) \quad \text{(Griffiths pg 47}$$

where T is the relativistic energy; $T = m_0 \gamma c^2$

Next, use the fact that $p^\mu p_\mu$ is invariant to write

$$p^\mu p_\mu = -m_0^2 c^2 = -(p_0)^2 + \vec{p}\cdot\vec{p}$$

$$-m_0^2 c^2 = -\frac{T^2}{c^2} + (qE_0 t)^2 + (p_i)^2 + 0$$

$$\Rightarrow T^2 = \underbrace{m_0^2 c^4 + c^2 p_i^2}_{T_0^2} + (cqE_0 t)^2$$

$$T = \sqrt{T_0^2 + (cqE_0 t)^2}$$

where T_0 = the initial total energy of the particle (Jackson p

now use $\vec{p} = m_0 \gamma \vec{v}$, and $T = m_0 \gamma c^2$ to write

$$\vec{v} = \frac{\vec{p}}{m_0 \gamma} = \frac{\vec{p}c^2}{T}$$

thus, for the x-component, one gets

$$\boxed{v_x(t) = \frac{dx}{dt} = \frac{p_x c^2}{T} = \frac{c^2 qE_0 t}{\sqrt{T_0^2 + (cqE_0 t)^2}}} \Rightarrow \boxed{\lim_{t\to\infty} v_x(t) = c}$$

$u = cqE_0 t \qquad du = cqE_0 dt \Rightarrow dt =$

$$\int dx = \int \frac{c^2 q E_0 t}{\sqrt{T_0^2 + (c q E_0 t)^2}} dt \Rightarrow \int \frac{\cancel{c}\, u}{\sqrt{T_0^2+u^2}} \frac{du}{\cancel{c} q E_0}$$

apply another substitution $T_0^2 + u^2 = \alpha \qquad d\alpha = 2u\,du \Rightarrow du = \frac{d\alpha}{2u}$.

$$\frac{1}{qE_0}\int \alpha^{-1/2} \cancel{u} \frac{d\alpha}{2\cancel{u}} = \frac{1}{2qE_0}\left(2\alpha^{1/2}\right) = \frac{1}{qE_0}\sqrt{T_0^2+(cqE_0t)^2} + \text{constant}$$

so $$\boxed{x(t) = \frac{1}{qE_0}\sqrt{T_0^2+(cqE_0t)^2} - \underbrace{T_0/qE_0}_{\text{to match } x(0)=0}}$$

for $v_y(t)$ one gets:

$$\boxed{\frac{dy}{dt} = \frac{p_y c^2}{T} = \frac{p_0 c^2}{\sqrt{T_0^2+(cqE_0t)^2}}} \Rightarrow \boxed{\lim_{t\to\infty} v_y(t) = 0}$$

make substitution $u = \frac{cqE_0t}{T_0} \qquad du = \frac{cqE_0}{T_0}dt \Rightarrow dt = \frac{T_0}{cqE_0}du$

so integral becomes

$$\int dy = \frac{p_0 c^{\cancel{2}} \cancel{T_0}}{\cancel{T_0} \cancel{c} qE_0}\int \frac{1}{\sqrt{1+u^2}}du = \frac{p_0 c}{qE_0}\sinh^{-1}(u)$$

thus $$\boxed{y(t) = \frac{p_0 c}{qE_0}\sinh^{-1}\left(\frac{cqE_0t}{T_0}\right)}$$

the motion is confined to the x-y plane, so $\boxed{v_z = 0 \quad z(t) = 0}$

b) find the trajectory $x(y)$.

eliminate t by taking $\sinh\left[\frac{qE_0 y}{p_0 c}\right] = \frac{cqE_0 t}{T_0}$

so $t = \frac{T_0}{cqE_0}\sinh\left[\frac{qE_0 y}{p_0 c}\right]$ ← plug this into $x(t)$ to yield

$$x(y) = \frac{1}{qE_0}\sqrt{T_0^2 + T_0^2\sinh^2\left(\frac{qE_0 y}{p_0 c}\right)} - T_0/qE_0$$

$$\boxed{x(y) = \frac{T_0}{qE_0}\left[\cosh\left(\frac{qE_0 y}{p_0 c}\right) - 1\right]}$$