Consider the motion of a charge in an electic field $\vec{E}$, using the correct equations of special relativity. Let $\vec{E} = E_0 \hat{x}$, with E_0 constant. Consider a particle with initial conditions

$$x = y = z = v_x = v_z = 0 \qquad \text{at } t=0$$
$$v_y \neq 0 \qquad \text{at } t=0$$

a) Solve for $v_x(t)$, $v_y(t)$, and $v_z(t)$. Also find simplified expressions for v_x and v_y when $t \to \infty$.

From Newton's 2nd law, we have

$$\frac{d\vec{p}}{dt} = \vec{F} = q\vec{E} + q\,\cancel{\vec{v}\times\vec{B}}^{\,0} = q\vec{E} = qE_0\hat{x} \qquad (1)$$

where p_μ is the 4-momentum defined a-like so

$$p_\mu = p_0 + \vec{p} = (E, p_x, p_y, p_z) \qquad \text{note: } c=1 \text{ here} \qquad (2)$$

From eq (1), we have

$$\frac{dp_x}{dt} = qE_0 \quad , \quad \frac{dp_y}{dt} = 0 = \frac{dp_z}{dt}$$

$$\Rightarrow p_x = qE_0 t \quad , p_y = c_1 \quad , p_z = c_2$$

from initial conditions, those values become

$$p_x = qE_0 t \quad , p_y = p_{yi} \quad , p_z = \emptyset$$

From Abers eq (12.76), we have, in natural units,

$$\vec{v} = \frac{\vec{p}}{E} = \frac{\vec{p}}{\sqrt{p^2 + m^2}} \qquad (3)$$

But, what is p^2?

$$p^2 = \vec{p}\cdot\vec{p} = p_x p^x + p_y p^y + p_z p^z = (qE_0 t)^2 + p_{y_i}^2 + 0 \qquad (4)$$

substituting this result into eq. (3) yields

$$\vec{v} = \frac{\vec{p}}{\sqrt{(qE_0 t)^2 + p_{y_i}^2 + m^2}}$$

note: $\sqrt{p_{y_i}^2 + m^2}$ is the initial energy, ε_0

So, we have

$$\boxed{\vec{v} = \frac{\vec{p}}{\sqrt{\varepsilon_0^2 + (qE_0t)^2}}}$$

Now, we can immediately writed down v_x, v_y, and v_z

$$v_x = \frac{p_x}{\sqrt{\varepsilon_0^2 + (qE_0t)^2}} = \frac{qE_0t}{\sqrt{\varepsilon_0^2 + (qE_0t)^2}} = \boxed{\frac{1}{\sqrt{1 + \left(\frac{\varepsilon_0}{qE_0t}\right)^2}}}$$

$$\Rightarrow \lim_{t\to\infty} v_x = 1$$

$$v_y = \frac{p_{y_i}}{\sqrt{\varepsilon_0^2 + (qE_0t)^2}} = \frac{p_{y_i}}{qE_0t} \frac{1}{\sqrt{1 + \left(\frac{\varepsilon_0}{qE_0t}\right)^2}} = \boxed{\frac{p_{y_i}}{qE_0t} v_x}$$

$$\Rightarrow \lim_{t\to\infty} v_y = 0$$

$$\boxed{v_z = 0}$$

(b) Find $x(t)$, $y(t)$, and Find the trajectory $x(y)$.

(i) $x(t)$

From part (a), we have

$$v_x = \frac{dx}{dt} = \frac{qE_0t}{\sqrt{\varepsilon_0^2 + (qE_0t)^2}} \Rightarrow x(t) = \int_0^t \frac{qE_0t'}{\sqrt{(qE_0t')^2 + \varepsilon_0^2}} dt'$$

let $u = qE_0t \Rightarrow du = qE_0dt$

So, we have

$$x(t) = \frac{1}{qE_0}\int_0^{qE_0t} \frac{u\,du}{\sqrt{u^2 + \varepsilon_0^2}} = \frac{1}{qE_0}\left[(u^2 + \varepsilon_0^2)^{1/2}\right]_0^{qE_0t}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{qE_0}\left[\sqrt{(qE_0t)^2 + \mathcal{E}_0^2} - \mathcal{E}_0\right]} \qquad (5)$$

note: $x(0) = \frac{1}{qE_0}\left[\mathcal{E}_0 - \mathcal{E}_0\right] = 0$ as required by initial conditions

(ii) $y(t)$

from part (a), we know

$$v_y = \frac{dy}{dt} = \frac{p_{yi}}{\sqrt{\mathcal{E}_0^2 + (qE_0t)^2}} \Rightarrow y(t) = \int_0^t \frac{p_{yi}\,dt'}{\sqrt{\mathcal{E}_0^2 + (qE_0t')^2}}$$

once again
let $u = qE_0t$

so,

$$y(t) = \frac{1}{qE_0}\int_0^{qE_0t} \frac{p_{yi}\,du}{\sqrt{\mathcal{E}_0^2 + u^2}} = \frac{p_{yi}}{qE_0}\int_0^{qE_0t} \frac{du}{\sqrt{\mathcal{E}_0^2 + u^2}}$$

$$= \frac{p_{yi}}{qE_0}\left[\sinh^{-1}\left(\frac{u}{\mathcal{E}_0}\right)\right]_0^{qE_0t} = \frac{p_{yi}}{qE_0}\left[\sinh^{-1}\left(\frac{qE_0t}{\mathcal{E}_0}\right) - \cancelto{0}{\sinh^{-1}(0)}\right]$$

$$\Rightarrow \boxed{y(t) = \frac{p_{yi}}{qE_0}\sinh^{-1}\left(\frac{qE_0t}{\mathcal{E}_0}\right)} \qquad (6)$$

note: $y(0) = \frac{p_{yi}}{qE_0}\sinh^{-1}(0) = 0$ as required by initial conditions

(iii) $x(y)$

solve eq. (5) for t:

$$(qE_0t)^2 + \mathcal{E}_0^2 = \left(qE_0x(t) + \mathcal{E}_0\right)^2$$

$$\Rightarrow \quad qE_0 t = \sqrt{(qE_0 x(t) + \varepsilon_0)^2 - \varepsilon_0^2}$$

$$\therefore \quad \boxed{t = \frac{1}{qE_0}\sqrt{(qE_0 x(t) + \varepsilon_0)^2 - \varepsilon_0^2}} \quad (7)$$

Now, solve eq (6) for t:

$$\frac{qE_0 t}{\varepsilon_0} = \sinh\left[\frac{qE_0 y(t)}{p_{yi}}\right]$$

$$\therefore \quad \boxed{t = \frac{\varepsilon_0}{qE_0}\sinh\left[\frac{qE_0 y(t)}{p_{yi}}\right]} \quad 8$$

Now, set eq (7) equal to eq (8) and solve for $x(t)$.

$$\frac{1}{qE_0}\sqrt{(qE_0 x(t) + \varepsilon_0)^2 - \varepsilon_0^2} = \frac{\varepsilon_0}{qE_0}\sinh\left[\frac{qE_0 y(t)}{p_{yi}}\right]$$

$$\Rightarrow \quad (qE_0 x(t) + \varepsilon_0)^2 - \varepsilon_0^2 = \varepsilon_0^2 \sinh^2\left[\frac{qE_0 y(t)}{p_{yi}}\right]$$

$$\Rightarrow \quad qE_0 x(t) + \varepsilon_0 = \sqrt{\varepsilon_0^2\left[\sinh^2\left(\frac{qE_0 y(t)}{p_{yi}}\right) + 1\right]}$$

$$= \sqrt{\varepsilon_0^2 \cosh^2\left(\frac{qE_0 y(t)}{p_{yi}}\right)} = \varepsilon_0 \cosh\left(\frac{qE_0 y(t)}{p_{yi}}\right)$$

$$\Rightarrow \quad x(t) = \frac{1}{qE_0}\left[\varepsilon_0\left(\cosh\left(\frac{qE_0 y(t)}{p_{yi}}\right) - 1\right)\right]$$

$$\therefore \quad \boxed{x(y) = \frac{\varepsilon_0}{qE_0}\left[\cosh\left(\frac{qE_0 y(t)}{p_{yi}} - 1\right)\right]}$$