Spring 1998 #9

Consider two particles, one spin ½ with spin $\vec{S}_1$, and one spin ½ with spin $\vec{S}_2$. They interact with a spin-spin interaction given by

$$H = \frac{4A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2$$

where A is a constant

a) Find the energy eigenvalues and the degeneracy of each level.

→ from addition of angular momentum, we find that the total spin lies between $|\vec{S}_1 - \vec{S}_2|$ and $(\vec{S}_1 + \vec{S}_2)$, so for two spin ½ particle the possibilities are 1 and 0. There will be four states, three with $s=1$ and one with $s=0$ (see Griffiths pg 166)

$S=1$ states : $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right]$, $|\downarrow\downarrow\rangle$ } triplet

$s=0$ state : $\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$ } singlet

to find the eigenvalues, first write $S^2 = (S_1+S_2)^2 = S_1^2 + 2S_1\cdot S_2 + S_2^2$

solving for $S_1 \cdot S_2$ yields

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)$$

the Hamiltonian can thus be written as

$$H = \frac{2A}{\hbar^2}\left(S^2 - S_1^2 - S_2^2\right)$$

we know the eigenvalues of S^2 are $s(s+1)\hbar^2$ which yields eigenvalues:

$$\frac{2A}{\hbar^2}\hbar^2\left[1(1+1) - \tfrac{3}{4} - \tfrac{3}{4}\right] = A \quad \text{for } S=1 \text{ states}$$

$$\frac{2A}{\hbar^2}\hbar^2\left[0(0+1) - \tfrac{3}{4} - \tfrac{3}{4}\right] = -3A \quad \text{for } S=0 \text{ state}$$

$$\boxed{\text{energy eigenvalues are: } A \text{ , 3 fold degeneracy};\quad -3A \text{ , no degeneracy}}$$

(b) Suppose the Hamiltonian is modified by adding a term $\frac{2\lambda}{\hbar} S_{2z}$ where λ is a constant and S_{2z} is the z component of $\vec{S}_2$. How does this change the energy levels and their degeneracy?

→ for states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ the shift is easy to determine:

for $|\uparrow\uparrow\rangle$ shift $= +\lambda$

for $|\downarrow\downarrow\rangle$ shift $= -\lambda$

for states $\frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]$ and $\frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$ we must construct the Hamiltonian matrix. Choose $|1\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]$ as basis vec

$|2\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$

$$H'|1\rangle = \frac{1}{\sqrt{2}}\left[-\lambda|\uparrow\downarrow\rangle + \lambda|\downarrow\uparrow\rangle\right] = \lambda|2\rangle \Rightarrow H' = \begin{pmatrix} \langle 1|H'|1\rangle & \langle 1|H'|2\rangle \\ \langle 2|H'|1\rangle & \langle 2|H'|2\rangle \end{pmatrix} = \begin{pmatrix} 0 & \\ \lambda & \end{pmatrix}$$

$$H'|2\rangle = \frac{1}{\sqrt{2}}\left[-\lambda|\uparrow\downarrow\rangle - \lambda|\downarrow\uparrow\rangle\right] = -\lambda|1\rangle$$

whereas

$$H_0|1\rangle = A|1\rangle \Rightarrow H_0 = \begin{pmatrix} A & 0 \\ 0 & -3A \end{pmatrix}$$

$$H_0|2\rangle = -3A|2\rangle$$

So $$H = H_0 + H' = \begin{pmatrix} A & -\lambda \\ \lambda & -3A \end{pmatrix}$$

Solving for eigenvalues gives new energies

$$\det\begin{pmatrix} A-E & -\lambda \\ \lambda & -3A-E \end{pmatrix} = 0 \Rightarrow (A-E)(-3A-E) + \lambda^2 = 0$$

$$E^2 + 2AE + \lambda^2 - 3A^2 = 0$$

$$E = -A \pm \sqrt{4A^2 - \lambda^2}$$

Thus, the new energies are

$A + \lambda$

$A - \lambda$

$-A + \sqrt{4A^2 - \lambda^2}$

$-A - \sqrt{4A^2 - \lambda^2}$