(a) 2 spin $\frac{1}{2}$ particles w/ $H = \frac{4A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$

so possible states are: $|\uparrow\uparrow\rangle \Leftrightarrow |s=1, m=1\rangle$

$|\downarrow\downarrow\rangle \Leftrightarrow |s=1, m=0\rangle$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \Leftrightarrow |s=1, m=-1\rangle$

and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \Leftrightarrow |s=0, m=0\rangle$

now note that $S^2 = (\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\vec{S}_1 \cdot \vec{S}_2$

so that $2\vec{S}_1 \cdot \vec{S}_2 = S^2 - S_1^2 - S_2^2$

so, applying H to these states gives:

$$H|\uparrow\uparrow\rangle = \frac{2A}{\hbar^2}(S^2 - S_1^2 - S_2^2)|\uparrow\uparrow\rangle \Rightarrow 2A\left(1(1+1) - \tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)$$
$$= 2A\left(2 - \tfrac{3}{4} - \tfrac{3}{4}\right) = A$$

$$H|\downarrow\downarrow\rangle \Rightarrow 2A\left(2 - \tfrac{3}{4} - \tfrac{3}{4}\right) = A$$

$$H\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{2A}{\sqrt{2}}\left[\left(2 - \tfrac{3}{4} - \tfrac{3}{4}\right)|\uparrow\downarrow\rangle + \left(2 - \tfrac{3}{4} - \tfrac{3}{4}\right)|\downarrow\uparrow\rangle\right] \Rightarrow \boxed{A}$$

$$H\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{2A}{\sqrt{2}}\left[\left(0(0+1) - \tfrac{3}{4} - \tfrac{3}{4}\right)|\uparrow\downarrow\rangle - \left(0(0+1) - \tfrac{3}{4} - \tfrac{3}{4}\right)|\downarrow\uparrow\rangle\right]$$
$$= -\tfrac{6}{4}(2A)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \Rightarrow \boxed{-3A}$$

so $E = -3A$

and $E = A$, 3-fold degeneracy

(b) new $H = \frac{4A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2 + \frac{2\lambda}{\hbar} S_{2z}$

now $H|\uparrow\uparrow\rangle = A|\uparrow\uparrow\rangle + \frac{2\lambda}{\hbar} S_{2z}|\uparrow\uparrow\rangle = A|\uparrow\uparrow\rangle + \frac{2\lambda}{\hbar}\frac{1}{2}\hbar|\uparrow\uparrow\rangle = (A+\lambda)|\uparrow\uparrow\rangle$

$H|\downarrow\downarrow\rangle = (A - \lambda)|\downarrow\downarrow\rangle$

but the other 2 states are not eigenstates of H

lets write the Hamiltonian matrix using the two remaining states as a basis:

$$|1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \qquad |2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

then $H|1\rangle = A|1\rangle + \lambda \frac{1}{\sqrt{2}}(-|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = A|1\rangle - \lambda|2\rangle$

and $H|2\rangle = A|2\rangle + \lambda \frac{1}{\sqrt{2}}(-|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = -3A|1\rangle - \lambda|1\rangle$

since $H|\text{arbitrary state}\rangle = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha H_{11} + \beta H_{12} \\ \alpha H_{21} + \beta H_{22} \end{pmatrix}$

and keeping in mind $H_0 = \begin{pmatrix} A & 0 \\ 0 & -3A \end{pmatrix}$, we have $H_{11} = A$, $H_{22} = -3A$
$H_{21} = H_{12} = -\lambda$

so we have $H = \begin{pmatrix} A & -\lambda \\ -\lambda & -3A \end{pmatrix}$

the eigenvalue equation is $H|n\rangle = E_n|n\rangle \Rightarrow (H - E_n)|n\rangle = 0$

$$\text{so } 0 = \begin{vmatrix} A - E_n & -\lambda \\ -\lambda & -3A - E_n \end{vmatrix} = (A - E_n)(-3A - E_n) - \lambda^2 = -3A^2 + 3AE_n - AE_n + E_n^2 - \lambda^2$$

$$= E_n^2 + 2AE_n + (-\lambda^2 - 3A^2)$$

$$\Rightarrow E_n = \frac{-2A \pm \sqrt{4A^2 + 4(\lambda^2 + 3A^2)}}{2}$$

$$\boxed{= -A \pm \sqrt{4A^2 + \lambda^2}}$$

So the new $E = \begin{cases} A + \lambda \\ A - \lambda \\ -A + \sqrt{4A^2 + \lambda^2} \\ -A - \sqrt{4A^2 + \lambda^2} \end{cases}$