Solution Problem #10
Fall 1999 Comps

Lea Fredrickson

10) (Quantum Mechanics) The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

where

$$H|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$$
$$H|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some angle θ. If it is known that a neutrino was definitely a $|\nu_\mu\rangle$ when it was produced, what is the probability of detecting a $|\nu_e\rangle$ after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $(1 - v/c)$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 \equiv m_1^2 - m_2^2$.

(This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment last year. In reality there is a third neutrino $|\nu_\tau\rangle$.)

let $\hbar = 1$

$$|\Psi_0\rangle = |\nu_\mu\rangle$$

$$|\Psi(t)\rangle = e^{-iHt}|\nu_\mu\rangle = e^{-iHt}|\Psi_0\rangle$$

$$|\Psi(t)\rangle = e^{-iHt}\left(-\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle\right)$$

$$|\Psi(t)\rangle = -e^{-i\sqrt{p^2c^2+m_1^2c^4}\,t}\sin\theta|\nu_1\rangle + e^{-i\sqrt{p^2c^2+m_2^2c^4}\,t}\cos\theta|\nu_2\rangle$$

The probability is given by

$$P(t) = |\langle\nu_e|\Psi(t)\rangle|^2$$

<u>let's call</u>

$$E_1 = \sqrt{p^2c^2 + m_1^2c^4}$$

$$E_2 = \sqrt{p^2c^2 + m_2^2c^4}$$

$$P(t) = \left| \left(\cos\theta |\nu_1\rangle + \sin\theta |\nu_2\rangle\right)\left(-\sin\theta e^{-iE_1t}|\nu_1\rangle + \cos\theta e^{-iE_2t}|\nu_2\rangle\right)\right|^2$$

$$P(t) = \left| \cos\theta \sin\theta \left(e^{-iE_2t} - e^{-iE_1t}\right)\right|^2$$

$$P(t) = \cos^2\theta \sin^2\theta \left[e^{-iE_2t} - e^{-iE_1t}\right]\left[e^{iE_2t} - e^{iE_1t}\right]$$

$$P(t) = \cos^2\theta \sin^2\theta \left[1 - e^{-i(E_1-E_2)t} - e^{i(E_1-E_2)t} + 1\right]$$

$$P(t) = \cos^2\theta \sin^2\theta \left[2 - 2\cos((E_1-E_2)t)\right]$$

now use approximations

$$m_1c \ll p \quad m_2c \ll p \quad \Rightarrow L = ct$$

$$(E_1 - E_2) = pc\left[\sqrt{1 + \frac{m_1^2c^2}{p^2}} - \sqrt{1 + \frac{m_2^2c^2}{p^2}}\right]$$

$$(E_1 - E_2) \approx pc\left[1 + \frac{m_1^2c^2}{2p^2} - 1 - \frac{m_2^2c^2}{2p^2}\right]$$ to first order...

$$(E_1 - E_2) \approx pc\left[\frac{c^2}{2p^2}\left(m_1^2 - m_2^2\right)\right] = \frac{c^3}{2p}\Delta m^2$$

Therefore,

$$P(t) = \cos^2\theta \sin^2\theta \left[2 - 2\cos\left(\frac{c^3 \Delta m^2 t}{2p}\right)\right]$$

$$P(t) = \frac{1}{2}\sin^2 2\theta \left[1 - \cos\left(\frac{c^3 \Delta m^2 t}{2p}\right)\right]$$

After some distanke L the time is $t = L/c$

$$P(L/c) = \frac{1}{2}\sin^2 2\theta \left[1 - \cos\left(\frac{L c^2 \Delta m^2}{2p}\right)\right]$$

is the probability for finding the neutrino in the electron neutrino state $|\nu_e\rangle$.

To simplify,

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\Rightarrow \boxed{P(L/c) = \sin^2 2\theta \sin^2\left(\frac{L c^2 \Delta m^2}{4p}\right)}$$