Fall 1999 # 11 (p 1 of 3)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Let $|\Psi_n\rangle, n = 0,1,2,\ldots$ be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0|\Psi_0\rangle + c_1|\Psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

See Aber's Final to 221a question # 4 parts (b)-(d)!!

let $\hbar = 1$!

we are given that $\langle\phi|H|\phi\rangle = \omega$, $H = \omega\left(a^\dagger a + \frac{1}{2}\right)$ in terms of raising and lowering operators

$$\Rightarrow \omega = \omega\left(c_0^*\langle\Psi_0| + c_1^*\langle\Psi_1|\right)\left(N + \tfrac{1}{2}\right)\left(c_0|\Psi_0\rangle + c_1|\Psi_1\rangle\right)$$

$$= \omega\left(c_0^*\langle\Psi_0| + c_1^*\langle\Psi_1|\right)\left(\frac{c_0}{2}|\Psi_0\rangle + \frac{3}{2}c_1|\Psi_1\rangle\right)$$

$$= \omega\left(\frac{|c_0|^2}{2} + \frac{3}{2}|c_1|^2\right)$$

$$\Rightarrow |c_0|^2 + 3|c_1|^2 = 2 \qquad (1)$$

From normalization condition, we have $|c_0|^2 + |c_1|^2 = 1$ (2)

$$(1) - (2) = 2|c_1|^2 = 1 \Rightarrow \boxed{|c_1|^2 = \frac{1}{2}}$$

sub this result in eq (2) yields

$$\boxed{|c_0|^2 = \frac{1}{2}}$$

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1| e^{i\theta_1}$. Suppose further that, not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{\sqrt{\hbar}}{\sqrt{4m\omega}}$$

what is θ_1?

let $\hbar = 1$!

we know that x in terms of a and a^+ is $\quad x = \frac{1}{\sqrt{2m\omega}}(a + a^+)$

so,

$$x|\phi\rangle = \frac{1}{\sqrt{2m\omega}}\left[(a+a^+)c_0|\psi_0\rangle + (a+a^+)c_1|\psi_1\rangle\right]$$

$$= \frac{1}{\sqrt{2m\omega}}\left[c_0 \overbrace{\cancel{a|\psi_0\rangle}}^{0} + c_0 a^+|\psi_0\rangle + c_1 a|\psi_1\rangle + c_1 a^+|\psi_1\rangle\right]$$

already lowest state

note: $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$
$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$= \frac{1}{\sqrt{2m\omega}}\left[c_0|\psi_1\rangle + c_1|\psi_0\rangle + c_1\sqrt{2}|\psi_2\rangle\right]$$

$$= \frac{1}{\sqrt{2m\omega}}\left[c_0|\psi_1\rangle + c_1\left(|\psi_0\rangle + \sqrt{2}|\psi_2\rangle\right)\right]$$

$$\Rightarrow \langle\phi|x|\phi\rangle = \frac{1}{\sqrt{2m\omega}}\left[c_0^*\langle\psi_0| + c_1^*\langle\psi_1|\right]\left[c_0|\psi_1\rangle + c_1\left(|\psi_0\rangle + \sqrt{2}|\psi_2\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2m\omega}}\left[c_0^* c_1 + c_1^* c_0\right] = \frac{c_0|c_1|}{\sqrt{2m\omega}}\left[e^{i\theta_1} + e^{-i\theta_1}\right]$$

c_0 is real, positive

$c_1 = |c_1|\,e^{i\theta_1}$

$$\Rightarrow \langle \phi | x | \phi \rangle = \frac{2 c_0 |c_1|}{\sqrt{2m\omega}} \cos\theta_1 \Big|_{c_0 = c_1 = \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2m\omega}} \cos\theta_1$$

we are given that $\langle \phi | x | \phi \rangle = \frac{1}{\sqrt{4m\omega}}$. Equating the two results, we get

$$\frac{1}{\sqrt{4m\omega}} = \frac{1}{\sqrt{2m\omega}} \cos\theta_1$$

$$\Rightarrow \cos\theta_1 = \frac{1}{\sqrt{2}} \qquad \therefore \boxed{\theta_1 = \frac{\pi}{4}}$$

(c) Now suppose the system is in the state $|\phi\rangle$ described above at $t=0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

So, we have

$$|\psi(t)\rangle = \underbrace{c_0 e^{-iE_0 t}}_{\equiv c_0(t)} |\psi_0\rangle + \underbrace{c_1 e^{-iE_1 t}}_{\equiv c_1(t)} |\psi_1\rangle$$

From part (b), we know that

$$\langle \psi(t) | x | \psi(t) \rangle = \frac{1}{\sqrt{2m\omega}} \left[c_0^*(t)\, c_1(t) + c_1^*(t)\, c_0(t) \right]$$

$$= \frac{c_0 |c_1|}{\sqrt{2m\omega}} \left[e^{i\frac{\pi}{4}} e^{iE_0 t} e^{-iE_1 t} + e^{-i\frac{\pi}{4}} e^{iE_1 t} e^{-iE_0 t} \right]$$

$$= \frac{1}{2} \frac{1}{\sqrt{2m\omega}} \left[e^{-i\left[(E_1 - E_0)t - \frac{\pi}{4}\right]} + e^{i\left[(E_1 - E_0)t - \frac{\pi}{4}\right]} \right]$$

note: at $t=0$, we got the initial condition from part (b).

$$\therefore \boxed{\langle \psi(t) | x | \psi(t) \rangle = \frac{1}{\sqrt{2m\omega}} \cos\left(\omega t - \frac{\pi}{4}\right), \quad \omega' = E_1 - E_0 = \frac{\omega}{2} \cdot \frac{3\omega}{2} = \omega}$$