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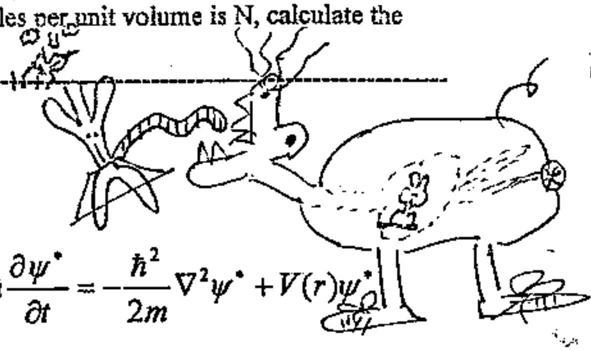
\# 12

Abeam of particles with uniform velocity v enters a region where some of them are absorbed. The absorption may be represented by introduction of a complex potential

$$V(\vec{r}) = V_R(\vec{r}) - iW_I(\vec{r})$$

where both V_R and W_I are real. If the number of absorbing particles per unit volume is N, calculate the cross section per particle for the absorption.



The probability density of states is given by

$$P(t) = \psi^*(t)\psi(t)$$

where ψ and ψ* satisfy the time dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi \qquad -i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V(r)\psi^*$$

Now use the relation $\frac{\partial P}{\partial t} = \left(\frac{\partial\psi^*}{\partial t}\right)\psi + \psi^*\left(\frac{\partial\psi}{\partial t}\right)$ and the Schrödinger equation to write

$$i\hbar\frac{\partial P}{\partial t} = \frac{\hbar^2}{2m}(\nabla^2\psi^*)\psi - V^*(r)\psi^*\psi - \frac{\hbar^2}{2m}(\nabla\psi)\psi^* + V(r)\psi\psi^*$$

$$i\hbar\frac{\partial P}{\partial t} = \frac{\hbar^2}{2m}\left[(\nabla^2\psi^*)\psi - (\nabla^2\psi)\psi^*\right] + \left[-V_R - iW_I + V_R - iW_I\right]\psi\psi^*$$

$$i\hbar\frac{\partial P}{\partial t} = \frac{\hbar^2}{2m}\nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi) - 2iW_I\psi\psi^*$$

For a particle traveling in one dimension, ψ takes the form

$$\psi(t) = e^{-iEt/\hbar}e^{ikx}$$

Thus,

$$\psi\nabla\psi^* - \psi^*\nabla\psi = e^{-iEt/\hbar}e^{ikx}(-ik)e^{iEt/\hbar}e^{-ikx} - e^{iEt/\hbar}e^{-ikx}(ik)e^{-iEt/\hbar}e^{ikx}$$

$$\psi\nabla\psi^* - \psi^*\nabla\psi = (-ik) - (ik) = 2ik$$

$$\nabla\cdot(2ik) = 0$$

Therefore, one can now write

$$i\hbar\frac{\partial P}{\partial t} = -2iW_I\psi\psi^* = -2iW_I P(t)$$

$$\frac{dP}{P} = \frac{-2W_I}{\hbar}dt$$

Solving for P (t) yields

$$P(t) = e^{-2W_I t/\hbar}$$

The function P (t) is the probability that a particle is not absorbed by time t. The probability that the particle is absorbed is therefore

$$P_{abs}(t) = 1 - e^{-2W_I t/\hbar}$$

If one now considers that the particle density is proportional to P(t), the flux will be proportional to the product P(t)v.

$$\text{Flux density} \propto P(t)v$$

Flux density has units of $\left(\#/\,Area \cdot \sec\right)$, and cross section is defined by the relation

(# events/sec)=(flux density)(cross section)

We also know that the number of absorption events between times 0 and t is proportional to $P_{abs}(t)$. After one second, the fraction of absorbed particles is given by

$$1 - e^{-2W_I/\hbar}$$

So the cross section can be found from

$$\left(1 - e^{-2W_I/\hbar}\right) = e^{2W_I/\hbar} v N \sigma$$

where σ is the absorption cross section per particle. Solving for σ yields

$$\sigma = \left(\frac{e^{2W_I/\hbar} - 1}{Nv}\right)$$

Note that if W_I=0, then the absorption cross section is zero, as should be expected. Also, the slower a particle moves, the larger its absorption cross section, which also makes intuitive sense.