Fall 1999 #13

The wavefunction $\Psi(t)$ of a stationary spin $\frac{1}{2}$ particle in a magnetic field $\vec{B}(t)$ obeys the time-dependent Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = -\vec{\sigma}\cdot\vec{B}(t)\Psi(t)$$

where $\vec{\sigma}$ is the vector containing the Pauli matrices.

Suppose $\vec{B}(t) = (b\cos(\omega t), b\sin(\omega t), B_0)$

a) Recall the pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Schrödinger's eq becomes

$$i\frac{\partial\Psi}{\partial t} = -\begin{pmatrix} B_0 & b\cos(\omega t) - ib\sin(\omega t) \\ b\cos(\omega t) + ib\sin(\omega t) & -B_0 \end{pmatrix}\Psi(t)$$

recall $e^{i\theta} = \cos\theta + i\sin\theta$

$$\boxed{i\frac{\partial\Psi(t)}{\partial t} = -\begin{pmatrix} B_0 & be^{-i\omega t} \\ be^{i\omega t} & -B_0 \end{pmatrix}\Psi(t)}$$

b) consider a solution of the form $\Psi(t) = \begin{pmatrix} f_+(t)e^{iB_0t} \\ f_-(t)e^{-iB_0t} \end{pmatrix}$

1st, we have

$$i\frac{\partial}{\partial t}\left(f_+(t)e^{iB_0t}\right) = -B_0f_+(t)e^{iB_0t} - bf_-(t)e^{-it(B_0+\omega)}$$

$$i\left[\left(\frac{\partial}{\partial t}f_+(t)\right)e^{iB_0t} + \cancel{f_+(t)(iB_0)e^{iB_0t}}\right] = \cancel{-B_0f_+(t)e^{iB_0t}} - bf_-(t)e^{-it(B_0+\omega)}$$

$$i\frac{\partial}{\partial t}f_+(t)e^{it(2B_0+\omega)} = -bf_-(t)$$

$$\boxed{i\frac{\partial f_+}{\partial t}e^{i\Omega t} = -bf_-(t)} \qquad \Omega = 2B_0+\omega$$

second, we have

$$i\frac{\partial}{\partial t}\left(f_-(t)e^{-iB_0t}\right) = -bf_+(t)e^{it(B_0+\omega)} + B_0f_-(t)e^{-iB_0t}$$

$$i\left[\left(\frac{\partial}{\partial t}f_-(t)\right)e^{-iB_0t} + \cancel{f_-(t)(-iB_0)e^{-iB_0t}}\right] = -bf_+(t)e^{it(B_0+\omega)} + \cancel{B_0f_-(t)e^{-iB_0t}}$$

$$i\frac{\partial f_-}{\partial t} = -bf_+e^{it(2B_0+\omega)}$$

$$\boxed{i\frac{\partial f_-}{\partial t} = -bf_+e^{i\Omega t}}$$

c) $i\frac{\partial^2 f_-}{\partial t^2} = -b\frac{\partial}{\partial t}\left(f_+ e^{i\Omega t}\right) = \underbrace{-b\left(\frac{\partial}{\partial t}f_+\right)e^{i\Omega t}}_{=\frac{-b^2 f_-}{i}} \underbrace{-i\Omega b f_+ e^{i\Omega t}}_{=-i\frac{\partial f_-}{\partial t}}$

$-\left(i\frac{\partial^2 f_-}{\partial t^2} + \Omega\frac{\partial f_-}{\partial t} - \frac{b^2 f_-}{i}\right) = 0$

$$\boxed{\frac{\partial^2 f_-}{\partial t^2} - i\Omega\frac{\partial f_-}{\partial t} + b^2 f_- = 0}$$

d) Given that $\Psi(0) = \begin{pmatrix}1\\0\end{pmatrix}$, find f_- at subsequent times.

Characteristic eq for f_- is $r^2 - i\Omega r + b^2 = 0$

has solutions

$$r = \frac{1}{2}\left(i\Omega \pm \sqrt{-\Omega^2 - 4b^2}\right) = \frac{1}{2}\left(i\Omega \pm i\sqrt{\Omega^2+4b^2}\right)$$

thus,

$$f_-(t) = A\exp\left(\frac{t}{2}\left(i\Omega + i\sqrt{\Omega^2+4b^2}\right)\right) + B\exp\left(\frac{t}{2}\left(i\Omega - i\sqrt{\Omega^2+4b^2}\right)\right)$$

from initial condition $\Psi(0) = \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}f_+(0)\\f_-(0)\end{pmatrix} \Rightarrow \begin{matrix}f_+(0)=1\\f_-(0)=0\end{matrix}$

we find $A+B=0 \Rightarrow B=-A$

use condition $i\frac{\partial f_-}{\partial t} = -b f_+ e^{i\Omega t}$ to find A, B

$$-\frac{A}{2}\left(\Omega + \sqrt{\Omega^2+4b^2}\right) - \frac{B}{2}\left(\Omega - \sqrt{\Omega^2+4b^2}\right) = -b$$

$$\cancel{\frac{A}{2}\Omega} + \frac{A}{2}\sqrt{\Omega^2+4b^2} - \cancel{\frac{A}{2}\Omega} + \frac{A}{2}\sqrt{\Omega^2+4b^2} = b \Rightarrow A = -B = b/\sqrt{\Omega^2+4b^2}$$

Thus,

$$f_-(t) = \left(e^{\frac{i\Omega t}{2}}\right)\left(\frac{b}{\sqrt{\Omega^2+4b^2}}\right)\left[\exp\left(\frac{i}{2}\sqrt{\Omega^2+4b^2}\,t\right) - \exp\left(\frac{-i}{2}\sqrt{\Omega^2+4b^2}\,t\right)\right]$$

$$\boxed{f_-(t) = \left(e^{\frac{i\Omega t}{2}}\right)\left(\frac{b}{\sqrt{\Omega^2+4b^2}}\right)\left(2i\sin\left(\frac{1}{2}\sqrt{\Omega^2+4b^2}\,t\right)\right)}$$

particle is in pure up state when $\sin\left(\frac{1}{2}\sqrt{\Omega^2+4b^2}\,t\right) = 0$

these occur at times

$$t = 2\pi n/\sqrt{\Omega^2+4b^2}$$

for positive integer n (positive so $t>0$)

this time will be shortest for large Ω