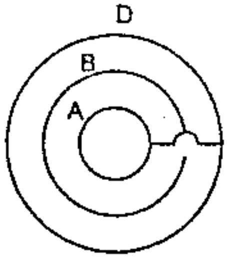
A capacitor is made of three thin concentric conducting spherical shells A, B, and D, with radii R_A, R_B, and R_D, respectively, as shown. Spheres A and D are connected by a fine insulated wire passing through a tiny hole in sphere B.

P37

Spring 1999 #1



a) Find the capacitance of the system.

To find capacitance, apply a charge $+Q_B$ to shell B and a charge $-Q_B$ to the combination of shells A and D. The capacitance is then given by the ratio C=Q/V where V is the potential difference between surfaces. Thus, to find the capacitance, do part (b) first and use result to find V.

b) Suppose a charge $-Q_B$ is placed on sphere B, and a net charge $+Q_B$ is placed on the sphere A/D system. How does the net charge $+Q_B$ distribute itself between the spheres A and D?

The charge $+Q_B$ will distribute itself so that shells A and D are at equal potential.
Recall that the potential due to a spherical shell of charge is:

$$V(r)=\begin{cases} Q/r_s & r\le r_s \\ Q/r & r>r_s \end{cases}$$

Where r_s is the radius of the shell.
One can now describe the potential in the four regions:

$$\frac{Q_A}{r_A}-\frac{Q_B}{r_B}+\frac{Q_D}{r_D}=V(r) \qquad r\le r_A$$

$$\frac{Q_A}{r}-\frac{Q_B}{r_B}+\frac{Q_D}{r_D}=V(r) \qquad r_A\le r\le r_B$$

$$\frac{Q_A}{r}-\frac{Q_B}{r}+\frac{Q_D}{r_D}=V(r) \qquad r_B\le r\le r_D$$

$$\frac{Q_A}{r}-\frac{Q_B}{r}+\frac{Q_D}{r}=V(r) \qquad r_D\le r$$

We want $V(r=r_A)=V(r=r_D)$. This condition yields:

$$\frac{Q_A}{r_A}-\frac{Q_B}{r_B}+\frac{Q_D}{r_D}=\frac{Q_A}{r_D}-\frac{Q_B}{r_D}+\frac{Q_D}{r_D}$$

$$Q_A\left(\frac{1}{r_A}-\frac{1}{r_D}\right)=Q_B\left(\frac{1}{r_B}-\frac{1}{r_D}\right)$$

$$Q_A\left(\frac{r_D-r_A}{r_Ar_D}\right)=Q_B\left(\frac{r_D-r_B}{r_Br_D}\right)$$

$$Q_A = Q_B \frac{r_A(r_D - r_B)}{r_B(r_D - r_A)}$$

Now we know that $Q_A + Q_D = Q_B$, so we can find Q_D:

$$Q_D = Q_B\left[1 - \frac{r_A(r_D - r_B)}{r_B(r_D - r_A)}\right]$$

$$Q_D = Q_B\left[\frac{r_B(r_D - r_A) - r_A(r_D - r_B)}{r_B(r_D - r_A)}\right]$$

$$Q_D = Q_B \frac{r_D(r_B - r_A)}{r_B(r_D - r_A)}$$

$\frac{Q_A}{r_A} - \frac{Q_A}{r_B}$ ← not Q_B?

Now we can finish part (a).

$$C = \frac{Q_B}{V_A - V_B}.$$

$$V_A - V_B = Q_A\left(\frac{r_B - r_A}{r_A r_B}\right)$$

Thus,

$$C = \frac{Q_B}{Q_B \frac{r_A(r_D - r_B)(r_B - r_A)}{r_B(r_D - r_A)r_A r_B}}$$

Simplification leads to:

$$C = \frac{r_B^{\,2}(r_D - r_A)}{(r_D - r_B)(r_B - r_A)}$$