An electron is at rest in a constant magnetic field $\vec{B}$ pointing in the z direction. The Hamiltonian is:

Spring 1999 #11

$$H = -\vec{\mu}\cdot\vec{B} = g\mu_0 \frac{\vec{S}}{\hbar}\cdot\vec{B}$$

where $\vec{B} = B_0\hat{z}$. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of $\vec{S}_z$ with eigenvalues $\pm\hbar/2$.

a) What are the eigenstates of H, and what is the energy difference between them?

$$H = \frac{g\mu_0}{\hbar}\frac{\hbar}{2}B_0\sigma_z = \frac{1}{2}g\mu_0 B_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

find eigenvalues of σ_z: $\det\begin{pmatrix}1-\lambda & 0\\ 0 & -1-\lambda\end{pmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) = 0$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$\rightarrow$ the eigenvalues of H are $\pm\frac{1}{2}g\mu_0 B_0$

find the eigenstates of σ_z

for $\lambda = 1$: $\begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow$ eigenstate $= \begin{pmatrix}1\\ 0\end{pmatrix}$

for $\lambda = -1$: $\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow$ eigenstate $= \begin{pmatrix}0\\ 1\end{pmatrix}$

Thus the eigenstates of H are:

$\boxed{\begin{pmatrix}1\\ 0\end{pmatrix} \text{ with energy } +\frac{1}{2}g\mu_0 B_0, \quad \begin{pmatrix}0\\ 1\end{pmatrix} \text{ with energy } -\frac{1}{2}g\mu_0 B_0 \quad \text{the difference in energy is } g\mu_0 B_0}$

b) At time $t=0$, the electron is in an eigenstate of S_x with eigenvalue $+\hbar/2$. Calculate $|\psi(t)\rangle$.

find eigenstate of $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ for $\lambda = +1$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} = -a + b = 0 \Rightarrow \text{eigenstate } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

thus $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\psi_+\rangle + \frac{1}{\sqrt{2}}|\psi_-\rangle$

So $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle = e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}\right)|\psi_+\rangle + e^{-iHt/\hbar}\left(\frac{1}{\sqrt{2}}\right)|\psi_-\rangle$

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(-i\frac{g\mu_0 B_0}{2\hbar}t\right)|\psi_+\rangle + \frac{1}{\sqrt{2}}\exp\left(+i\frac{g\mu_0 B_0}{2\hbar}t\right)|\psi_-\rangle}$$

c) For the state found in (b), what are the expectation values of the three components of spin at any time t?

$$\langle S_x\rangle = \langle \Psi(t)|S_x|\Psi(t)\rangle = \frac{\hbar}{4}\begin{pmatrix} e^{+i\alpha} & e^{-i\alpha}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} e^{-i\alpha} \\ e^{+i\alpha}\end{pmatrix}$$

$$\langle S_x\rangle = \frac{\hbar}{4}\begin{pmatrix} e^{+i\alpha} & e^{-i\alpha}\end{pmatrix}\begin{pmatrix} e^{+i\alpha} \\ e^{-i\alpha}\end{pmatrix} = \frac{\hbar}{4}\left(e^{2i\alpha} + e^{-2i\alpha}\right) = \frac{\hbar}{2}\cos(2\alpha)$$

$$\boxed{\langle S_x\rangle = \frac{\hbar}{2}\cos\left(\frac{g\mu_0 B_0}{\hbar}t\right)}$$

$$\langle S_y\rangle = \frac{\hbar}{4}\begin{pmatrix} e^{+i\alpha} & e^{-i\alpha}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} e^{-i\alpha} \\ e^{i\alpha}\end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} e^{i\alpha} & e^{-i\alpha}\end{pmatrix}\begin{pmatrix} -ie^{i\alpha} \\ ie^{-i\alpha}\end{pmatrix}$$

$$\langle S_y\rangle = \frac{\hbar}{4}\left(-ie^{2i\alpha} + ie^{-2i\alpha}\right) = \frac{\hbar}{4}\left(-i\cancel{\cos(2\alpha)} - i^2\sin(2\alpha) + i\cancel{\cos(2\alpha)} - i^2 \sin\ldots\right.$$

$$\boxed{\langle S_y\rangle = \frac{\hbar}{2}\sin\left(\frac{g\mu_0 B_0}{\hbar}t\right)}$$

$$\langle S_z\rangle = \frac{\hbar}{4}\begin{pmatrix} e^{+i\alpha} & e^{-i\alpha}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} e^{-i\alpha} \\ e^{i\alpha}\end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} e^{i\alpha} & e^{-i\alpha}\end{pmatrix}\begin{pmatrix} e^{-i\alpha} \\ -e^{i\alpha}\end{pmatrix}$$

$$\langle S_z\rangle = \frac{\hbar}{4}\left(e^{i\alpha}e^{-i\alpha} - e^{-i\alpha}e^{i\alpha}\right)$$

$$\boxed{\langle S_z\rangle = 0}$$