An electron is at rest in a constant magnetic field $\vec{B}$ pointing along the z direction. The Hamiltonian is

$$H = -\vec{\mu}\cdot\vec{B} = g\mu_0 \frac{\vec{S}}{\hbar}\cdot\vec{B}$$

where $\vec{B} = B_0\hat{z}$, and $\hat{z}$ is the unit vector in the z direction. Since the electron is at rest, you can treat this as a two-state system. Let $|\Psi_\pm\rangle$ be the eigenstates of S_z with eigenvalues $\pm\hbar/2$ respectively.

(a) What are the eigenstates of the Hamiltonian, and what is the energy difference between them?

Let $\hbar = 1$!

So, the Hamiltonian is

$$H = g\mu_0 \vec{S}\cdot\vec{B} = \frac{g\mu_0}{2} B_0 \sigma_z \quad , \text{ since } \vec{B} = B_0\hat{z}$$

$$H = \frac{g\mu_0}{2}B_0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since this is a diagonal matrix, we immediately know that the eigenvalues are

$$\boxed{\lambda = \pm \frac{g\mu_0 B_0}{2}}$$

Since we are dealing with σ_z, we know the eigenstates are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

Let's show this.

$\lambda = +\frac{g\mu_0 B_0}{2}$

$$\begin{pmatrix} \frac{g\mu_0 B_0}{2} - \frac{g\mu_0 B_0}{2} & 0 \\ 0 & -\frac{g\mu_0 B_0}{2} + \frac{g\mu_0 B_0}{2} \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -g\mu_0 B_0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \phi_1 = 1,\ \phi_2 = 0 \qquad \therefore \boxed{\left|\lambda = +\tfrac{g\mu_0 B_0}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}}$$

$\lambda = -\frac{g\mu_0 B_0}{2}$

$$\begin{pmatrix} g\mu_0 B_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \phi_1 = 0,\ \phi_2 = 1$$

$$\therefore \boxed{\left|\lambda = -\tfrac{g\mu_0 B_0}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}} \longleftarrow \text{as expected!}$$

. So the energy difference, Δ, is

$$\boxed{\Delta = g\mu_0 B_0}$$

(b) At time $t=0$, the electron is in an eigenstate of S_x with eigenvalue $+\hbar/2$. Calculate $|\psi(t)\rangle$ for any t in terms of the constants and $|\psi_I\rangle$.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

we want to find eigenstate of $\lambda = +1$. So, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow \left.\begin{matrix} -\phi_1 + \phi_2 = 0 \\ \phi_1 - \phi_2 = 0 \end{matrix}\right\} \Rightarrow |\lambda = +1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now, write this eigenstate in terms of the eigenstates of σ_z.

$$|\psi(t=0)\rangle = |\lambda = +1\rangle = \frac{1}{\sqrt{2}}\left[|\psi_+\rangle + |\psi_-\rangle \right] \quad \text{where } |\psi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |\psi_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Applying the time evolution operator, we get ($\hbar = 1$)

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-iHt} |\psi_+\rangle + e^{-iHt} |\psi_-\rangle \right]$$

using the energies found in part (a), this becomes

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{\frac{-ig\mu_0 B_0 t}{2}} |\psi_+\rangle + e^{\frac{ig\mu_0 B_0 t}{2}} |\psi_-\rangle \right]}$$

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

$$\text{let } \alpha = \frac{g\mu_0 B_0 t}{2}$$

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(i) $\langle S_x \rangle$

$$\langle \psi(t) | S_x | \psi(t) \rangle = \langle \psi(t) | \frac{\sigma_x}{2} | \psi(t) \rangle = \frac{1}{4} \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} \\ e^{i\alpha} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} e^{i\alpha} \\ e^{-i\alpha} \end{pmatrix} = \frac{1}{4}\left(e^{i2\alpha} + e^{-i2\alpha} \right) = \frac{1}{2}\cos(2\alpha)$$

$$\therefore \boxed{\langle S_x \rangle = \frac{1}{2}\cos\left[g\mu_0 B_0 t \right]}$$

(ii) $\langle S_y \rangle$

$$\langle \psi(t) | \frac{\sigma_y}{2} | \psi(t) \rangle = \frac{1}{4} \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} \\ e^{+i\alpha} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} -i e^{+i\alpha} \\ i e^{-i\alpha} \end{pmatrix} = \frac{1}{4}\left(-i e^{i2\alpha} + i e^{-i2\alpha} \right)$$

$$= \frac{1}{4i}\left(e^{i2\alpha} - e^{-i2\alpha} \right) = \frac{1}{2}\sin(2\alpha)$$

$$\therefore \boxed{\langle S_y \rangle = \frac{1}{2}\sin\left[g\mu_0 B_0 t \right]}$$

(iii) $\langle S_z \rangle$

$$\langle \psi(t) | \frac{\sigma_z}{2} | \psi(t) \rangle = \frac{1}{4} \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} \\ e^{i\alpha} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} e^{-i\alpha} \\ -e^{i\alpha} \end{pmatrix} = \frac{1}{4}\left(1 - 1 \right) = 0$$

$$\therefore \boxed{\langle S_z \rangle = 0}$$