Spring 1999 #12

1-Dimensional Harmonic oscillator has the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

a) Use raising and lowering operators to work out 1st order differential equation for the ground state wave function.

refer to Abers sect 3.2.4

→ the ground state is defined by the property that $a|\psi_0\rangle = 0$

where a is the lowering operator:

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + ip) \qquad \text{(Abers 3.2.6)}$$

thus, in the x' basis $\langle x'|a|\psi_0\rangle = 0$

$$\langle x'|m\omega x + ip|\psi_0\rangle = 0$$

recall: $\overset{\text{operator}}{x}|\psi_0\rangle = \overset{\text{number}}{x}|\psi_0\rangle$

$p|\psi_0\rangle = -i\hbar\frac{d}{dx}|\psi_0\rangle$

$$m\omega\langle x'|x|\psi_0\rangle + i\langle x'|p|\psi_0\rangle = 0$$

$$m\omega x'\psi_0(x') + \hbar\frac{d}{dx'}\psi_0(x') = 0$$

now, one can drop the primes

$$\boxed{\frac{d}{dx}\psi_0(x) = -\frac{m\omega x}{\hbar}\psi_0(x)}$$

b) solve this equation to find the unnormalized ground state wave function.

→ the equation is separable, and can be rewritten as

$$\frac{d\psi_0(x)}{\psi_0(x)} = -\frac{m\omega x}{\hbar}dx$$

integration yields:

$$\ln\psi_0(x) = -\frac{m\omega x^2}{2\hbar} + \text{const}$$

$$\boxed{\psi_0(x) = A\, e^{-\frac{m\omega x^2}{2\hbar}}}$$

normalization constant

c) Use similar technique to find wave function of 1st excited st

→ apply raising operator to ground state $a^{+}|\psi_0\rangle = |\psi_1\rangle$

$a^{+} = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x - i$

$\psi_1(x') = \langle x'|a^{+}|\psi_0\rangle = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\langle x'|x|\psi_0\rangle - \hbar\langle x'|\frac{d}{dx}|\psi_0\rangle\right)$

$$\psi_1(x) = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\, x A e^{-\frac{m\omega}{2\hbar}x^2} - \cancel{\hbar}\left(\frac{-m\omega}{2\cancel{\hbar}}\right)A e^{-\frac{m\omega}{2\hbar}x^2}(2x)\right)$$

$$\boxed{\psi_1(x) = B x e^{-\frac{m\omega}{2\hbar}x^2}}$$ → where B is normalization constant

d) Define the operator $U = e^{-ipb/\hbar}$ for b = real #, p = momentum

Find the ground state wave function for the Hamiltonian

$$H = U H_0 U^{+}$$

→ want to find

$$\langle x|e^{-\frac{ibp}{\hbar}} H e^{\frac{ibp}{\hbar}}|\psi_0\rangle$$

recall Baker-Hausdorff Lemma:

$$e^{i\Theta B} A e^{-i\Theta B} = A + i\Theta[B,A] - \frac{\Theta^2}{2}[B,[B,A]] + \cdots$$

so, after absorbing (-) sign into b ($b' = -b$), one gets

$$\langle x|H|\psi_0\rangle + \langle x|i\tfrac{b'}{\hbar}[p,H]|\psi_0\rangle - \frac{1}{2}\left(\frac{b'}{\hbar}\right)^2\langle x|[p,[p,H]]|\psi_0\rangle$$

find $[p,H] = \left[p, \frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right] = \frac{1}{2m}\underbrace{[p,p^2]}_{\to 0} + \frac{m\omega^2}{2}[p,x^2]$

$\hookrightarrow = x[p,x] + [p,x]x$

$= -i\hbar x - i\hbar x = -2i$

Thus $[p,H] = -2i\hbar x$

and furthermore $[p,[p,H]] = -2i\hbar[p,x] = -2i\hbar(-i\hbar) = -2\hbar^2$

all higher commutators are zero

$$\langle x|H|\psi_0\rangle = \tfrac{1}{2}\hbar\omega\,\psi_0(x) - 2i^2 b' x\,\psi_0(x) - \tfrac{1}{2}(b')^2(-2)\,\psi_0(x)$$

$$\boxed{\langle x|H|\psi_0\rangle = \left(\tfrac{1}{2}\hbar\omega + b\right)\psi_0(x) - 2b\,x\,\psi_0(x)}$$ → where $\psi_0(x)$ was found in part (b).

$\omega/\sqrt{2}$

Why is this the ground state ψ_0? ... $U\psi_0 U^{+}$ ... ground state