Spring 1999 #13

A beam of particles scatters off an impenetrable sphere radius a. I.e, the potential is zero outside the sphe infinite inside. The wave fuction must therefore vanish at $r=a$. What is the total cross section in the limit of zero incident kinetic energy?

(see Griffith's QM pgs 361-362)

review this problem griffith

the potential $V(r)=\begin{cases}\infty & r\le a\\ 0 & r>a\end{cases}$ imposes boundary condition

$\Psi(a,\theta)=0$

the general solution for $\Psi(r,\theta,\phi)$ is

$$\Psi(r,\theta,\phi)=A\left\{e^{ikz}+\sum_{l,m}C_{l,m}h_l^{(1)}(kr)\,Y_l^m(\theta,\phi)\right\}$$

where e^{ikz} represents the incoming wave

$h_l^{(1)}(x)\equiv j_l(x)+i n_l(x)$ is the Hankel function of 1st kind

for azimuthal symmetry, $m=0$. We can also rewrite e^{ikz} as

$$e^{ikz}=\sum_{l=0}^{\infty}i^l(2l+1)\,j_l(kr)\,P_l(\cos\theta)$$

(A) thus $\Psi(a,\theta)=\sum_{l=0}^{\infty}\left[i^l(2l+1)\,j_l(ka)+\sqrt{\frac{2l+1}{4\pi}}\,C_l\,h_l^{(1)}(ka)\right]P_l(\cos\theta)=0$

comparison to the general solution $\Psi(r,\theta,\phi)=A\left\{e^{ikz}+f(\theta,\phi)\frac{e^{ikr}}{r}\right\}$ shows that the scattering amplite $f(\theta)$ is

$$f(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(-i)^{l+1}\sqrt{\frac{2l+1}{4\pi}}\,C_l\,P_l(\cos\theta)$$

the total cross section is thus $\quad\sigma=\frac{1}{k^2}\sum_{l=0}^{\infty}|C_l|^2$

Now solve (A) for C_l values, which gives,

$$C_l=-i^l\sqrt{4\pi(2l+1)}\,\frac{j_l(ka)}{h_l^{(1)}(ka)}$$

$\Rightarrow$

thus, the total cross section is

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left| \frac{j_l(ka)}{h_l^{(1)}(ka)} \right|^2$$

now write $\dfrac{j_l(ka)}{h_l^{(1)}(ka)} = \dfrac{j_l(ka)}{j_l(ka) + i n_l(ka)} \approx \dfrac{-i\, j_l(ka)}{n_l(ka)}$

(for small (ka), $n_l(ka) \gg j_l(ka)$)

for the low incident energy limit, we need only keep the 1st term $(l=0)$. Now using $j_0(ka) = \frac{\sin(ka)}{ka}$, $n_0(ka) = \frac{-\cos(ka)}{ka}$ one gets

$$\sigma = \frac{4\pi}{k^2} (2(0)+1) \left| -i \frac{\sin(ka)}{\cancel{ka}} \frac{\cancel{ka}}{(-\cos(ka))} \right|^2$$

$$\sigma = \frac{4\pi}{k^2} \sin^2(ka) \qquad (\cos(ka) \approx 1 \text{ for small } ka)$$

expand $\sin(ka)$ as Taylor series

$$\sigma = \frac{4\pi}{k^2} \left(ka - \frac{(ka)^3}{3!} + \cdots \right) \left(ka - \frac{(ka)^3}{3!} + \cdots \right)$$

since $ka \ll 1$ keep only first term of $\sin(ka)$ expansion to yield

$$\sigma = \frac{4\pi}{k^2} k^2 a^2$$

$$\boxed{\sigma = 4\pi a^2}$$