A beam of particles scatters off an impenetrable sphere of radius a. I.e., the potential is zero outside the sphere, and infinite inside. The wave function must therefore vanish at $r=a$. What is the total cross section in the limit of zero incident kinetic energy? (see Abers p283, 284)

So, we have

$$V(r) = V_0 \, \Theta(a-r) \qquad \text{in the limit } V_0 \to \infty.$$

The radial wave equation is ($\hbar = 1$)

$$u'' + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)u - \lambda u = 0$$

where $k^2 = 2mE$, $u = rR(r)$, $\lambda = 2mV(r)$ and the prime indicates $\frac{d}{dr}$.

For our case, we have

$$u'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]u = 0 \qquad r > a$$

$$0 \qquad r < a$$

The solution for $r > a$ is spherical Bessel functions and Neumann functions:

$$R_\ell(r) = a_\ell \, j_\ell(kr) + b_\ell \, n_\ell(kr)$$

Since the wave function must be continuous at $r = a$, we have

$$R_\ell(r=a) = 0 = a_\ell \, j_\ell(ka) + b_\ell \, n_\ell(ka)$$

$$\Rightarrow a_\ell \, j_\ell(ka) = -b_\ell \, n_\ell(ka) \qquad (1)$$

now, consider as $r \to \infty$

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\cos\left[kr - \frac{\pi}{2}(\ell+1)\right]}{kr} = \frac{\sin\left(kr - \frac{\pi\ell}{2}\right)}{kr}$$

and

$$n_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin\left[kr - \frac{\pi}{2}(\ell+1)\right]}{kr} = \frac{-\cos\left(kr - \frac{\pi\ell}{2}\right)}{kr}$$

we consider this region since it is easier and since we can invoke the continuity of the wave function.

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So, for large r, our wave function has the form

$$R_\ell(r) = \frac{a_\ell \sin\left(kr - \frac{\pi \ell}{2}\right)}{kr} - \frac{b_\ell \cos\left(kr - \pi\ell/2\right)}{kr}$$

it is convention to re-write this as follows:

$$R_\ell(r) = \frac{e^{i\delta_\ell}}{kr} \sin\left(kr - \frac{\pi \ell}{2} + \delta_\ell\right)$$

where $\tan \delta_\ell = -\frac{b_\ell}{a_\ell} = \frac{j_\ell(ka)}{n_\ell(ka)}$

I leave it to the reader to verify this .... and when you do, let me know how ☺. We are told to consider the region in which the incident kinetic region goes to zero. That is,

$$k \to 0 \Rightarrow \ell = 0$$

So, in this limit, the total cross section is given by Abers eq 8.121

$$\sigma_{tot}(0) = \frac{4\pi}{k^2} \sin^2 \delta_0$$

where

$$\tan \delta_0 = \frac{j_0(ka)}{n_0(ka)} = -\tan(ka) \Rightarrow \delta_0 = -ka$$

So,

$$\sigma_{tot}(0) = \frac{4\pi}{k^2} \sin^2(-ka) \xrightarrow[k\to 0]{} \frac{4\pi}{k^2}(-ka)^2$$

Thus,

$$\boxed{\sigma_{tot}(0) \to 4\pi a^2}$$

← note: For high energies $\sigma_\ell \simeq 0$

→ this is 4 times the classical cross sectional area. I think this is so because it includes the outward and inward scattering.