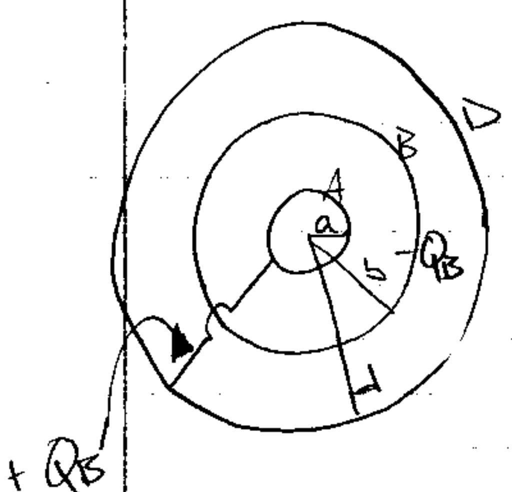
Sp 1999 #1



Concentric hollow conducting spheres A, B, D with radii a, b & d. A & D are shorted to each other

a) Capacitance?

b) if $-Q_B$ is placed on sphere B & $+Q_B$ on A & D system, how does the $+Q_B$ distribute over A & D?

Solution: Do the second part first. A & D must be @ equipotentials.

let $\phi(\infty) \to 0$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \quad \text{(in general)}$$

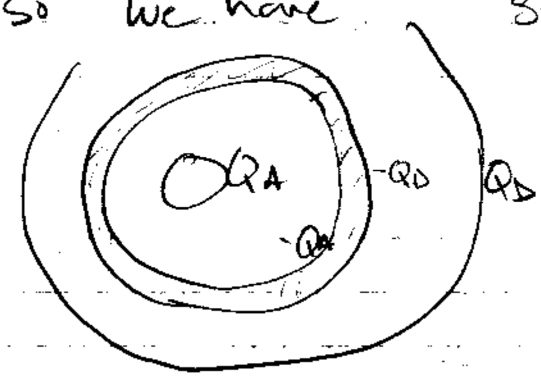
let Q_A be the charge that ends up on sphere A, likewise for Q_D & sphere D.

So $Q_A + Q_D = Q_B$

Therefore there will be $-Q_A$ induced on the inner side of B, and $+Q_A$ induced on the outer surface of B. There is also a charge $-Q_B$ on the outside of B, so the net charge on the outside of B is

$$-Q_B + Q_A = (-Q_A - Q_D) + Q_A = -Q_A.$$

So we have



So the total charge is

$$Q_A + (-Q_A) + (-Q_D) + Q_D = 0$$

therefore

$$E = \frac{Q_A}{4\pi\varepsilon_0 r^2} \quad a \leq r < b$$

$$E = \frac{-Q_D}{4\pi\varepsilon_0 r^2} \quad b < r < d$$

$$E = 0 \quad r < a,\ r > d$$

so $\phi = -\int_\infty^r \vec{E} \cdot d\vec{l}$

$$\phi(a) = -\int_\infty^d \vec{E} \cdot d\vec{l} - \int_d^b \vec{E} \cdot d\vec{l} - \int_b^a \vec{E} \cdot d\vec{l}$$

$$= -0 + \frac{Q_D}{4\pi\varepsilon_0} \int_d^b \frac{1}{r^2}\,dr - \frac{Q_A}{4\pi\varepsilon_0} \int_b^a \frac{1}{r^2}\,dr$$

$$= \overbrace{\frac{Q_D}{4\pi\varepsilon_0} \left[-\frac{1}{r} \right]_d^b}^{\phi(b)} - \frac{Q_A}{4\pi\varepsilon_0} \left[-\frac{1}{r} \right]_b^a$$

$$= \frac{Q_D}{4\pi\varepsilon_0} \left(-\frac{1}{b} + \frac{1}{d} \right) + \frac{Q_A}{4\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$\phi(d) = -\int_\infty^d \vec{E} \cdot d\vec{l} = 0$$

The inner & outer shells are shorted, so they have the same potential...

$$Q_A = Q_B - Q_D$$

so $\phi(a) = \phi(d) = 0$ gives

$$Q_D\left(-\frac{1}{b} + \frac{1}{d}\right) = -Q_A\left(\frac{1}{a} - \frac{1}{b}\right)$$

Find Q_A, Q_D

$$\boxed{Q_D\left(\frac{1}{d} - \frac{1}{b}\right) = Q_A\left(\frac{1}{b} - \frac{1}{a}\right)}$$

$$Q_D\left(\frac{1}{d} - \frac{1}{b}\right) = (Q_B - Q_D)\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$Q_D\left(\frac{1}{d} - \frac{1}{b}\right) + Q_D\left(\frac{1}{b} - \frac{1}{a}\right) = Q_B\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$Q_D\left(\frac{1}{d} - \frac{1}{a}\right) = Q_B\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$Q_D\left(\frac{a-d}{ad}\right) = Q_B\left(\frac{a-b}{ba}\right)$$

$$\boxed{Q_D = Q_B\left(\frac{d}{b}\right)\left(\frac{a-b}{a-d}\right)}$$

$$(Q_B - Q_A)\left(\frac{1}{d} - \frac{1}{b}\right) = Q_A\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$Q_B\left(\frac{b-d}{db}\right) = Q_A\left[\left(\frac{1}{d} - \frac{1}{b}\right) + \left(\frac{1}{b} - \frac{1}{a}\right)\right]$$

$$= Q_A\left(\frac{1}{d} - \frac{1}{a}\right)$$

$$= Q_A\left(\frac{a-d}{ad}\right)$$

$$\boxed{Q_A = Q_B\left(\frac{a}{b}\right)\left(\frac{b-d}{a-d}\right)}$$

Now the capacitance is ...

from the inner most shell to the intermediate shell:

$$C_{ab} = \frac{Q_A}{\phi(a)-\phi(b)} = \frac{Q_A}{-\phi(b)}$$

$$\phi(b) = \frac{Q_D}{4\pi\varepsilon_0}\left(\frac{1}{d}-\frac{1}{b}\right)$$

$$= \frac{Q_A}{4\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$= \frac{Q_A}{4\pi\varepsilon_0}\left(\frac{a-b}{ab}\right)$$

$$\boxed{C_{ab} = 4\pi\varepsilon_0 \frac{ab}{b-a}}$$

from the intermediate shell to the outer shell

$$C_{bd} = \frac{Q_D}{\phi(b)-\phi(d)} = \frac{Q_D}{\phi(b)} = 4\pi\varepsilon_0\left(\frac{bd}{b-d}\right)$$

So in series ...

$$C = \left[\frac{1}{C_{ab}} - \frac{1}{C_{bd}}\right]^{-1} = \left[\frac{C_{bd}-C_{ab}}{C_{ab}C_{bd}}\right]^{-1} = \frac{C_{ab}C_{bd}}{C_{bd}-C_{ab}}$$

$$C = \frac{(4\pi\varepsilon_0)^2\left(\frac{ab}{b-a}\right)\left(\frac{bd}{b-d}\right)}{(4\pi\varepsilon_0)\left[\frac{bd}{b-d} - \frac{ab}{b-a}\right]}$$

$$= \frac{4\pi\varepsilon_0 (ab^2d)\left(\frac{1}{b-a}\right)\left(\frac{1}{b-d}\right)}{\frac{bd(b-a) - ab(b-d)}{(b-d)(b-a)}}$$

This is the answer in Lim's book - but its wrong - I think.

$$= \frac{4\pi\varepsilon_0\, abd}{d(b-a) - a(b-d)} = \frac{4\pi\varepsilon_0\, abd}{db-ab} = 4\pi\varepsilon_0\left(\frac{ad}{d-a}\right)$$

Now the way in the solution

$$C = \frac{Q_B}{\phi(b)}$$

in parallel...

$$C = C_{ab} + C_{bd}$$

$$= 4\pi\varepsilon_0\, b\left(\frac{a}{b-a} - \frac{d}{b-d}\right)$$

$$= 4\pi\varepsilon_0\, b\left(\frac{a(b-d) - d(b-a)}{(b-a)(b-d)}\right)$$

$$= 4\pi\varepsilon_0\, b\left(\frac{ab-db}{(b-a)(b-d)}\right)$$

$$= 4\pi\varepsilon_0\, b^2\left(\frac{a-d}{(b-a)(b-d)}\right)$$ this is correct