[redacted] Spring 1999 # 3

a) Consider a charge q that sits at rest at point $(x_0, 0, 0)$ near a conducting surface, the plane at $x = 0$. Calculate the force on the charge.

→ The boundary condition can be met by placing an image charge $-q$ at position $(-x_0, 0, 0)$. The force between charges is then

$$F = \frac{-q^2}{4\pi\epsilon_0 (x_0 - (-x_0))^2}\hat{x}$$

$$\boxed{F = -\frac{q^2}{16\pi\epsilon_0 x_0^2}\hat{x}} \quad \text{(MKS units)}$$

b) Calculate the time T for the charge to hit the conductor. Ignore relativistic effects.

→ For a charge of mass m, the equation describing the charge's motion is

$$\ddot{x} = \frac{q^2}{16\pi\epsilon_0 x^2}(-\hat{x})$$

This equation is difficult to solve since the acceleration depends on the distance from the plane. However, by treating the acceleration as constant, one can write

$$\frac{1}{2}\ddot{x}t^2 = \frac{q^2}{32\pi\epsilon_0 x_0^2}t^2 = x_0$$

$$\Rightarrow \boxed{T = \frac{4\sqrt{2\pi\epsilon_0 x_0^3}}{q}}$$

the actual time taken will be less because the acceleration only gets larger as the charge gets closer to the plane.

c) Consider motion of charge when there is also $\vec{B}$ field $\vec{B} = B_0\hat{z}$ in the limit that B_0 is very large.

(See Griffith's QM pgs 199-201)

In this case we can consider the particle to be moving a combination of uniform $\vec{E}$ and $\vec{B}$ fields at right angles.
The electric field is:

$$\vec{E} = \frac{-q}{16\pi\epsilon_0 x_0^2}\hat{x} = E_0\hat{x}$$

magnetic field is

$$\vec{B} = B_0\hat{z}$$

The force is then given by the Lorentz force Law:

$$F = q\left[E + (v \times B)\right]$$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & B_0 \end{vmatrix} = \hat{x}(B_0\dot{y}) - \hat{y}(B_0\dot{x})$$

thus

$$F = (qE_0 + qB_0\dot{y})\hat{x} - (qB_0\dot{x})\hat{y}$$

for each component then:

$$\ddot{x} = (qE_0 + qB_0\dot{y})/m \qquad \ddot{y} = (-qB_0\dot{x})/m$$

define $\omega = \frac{qB}{m}$

$$\ddot{x} = \omega\left(\frac{E}{B} - \dot{y}\right) \qquad \ddot{y} = -\omega\dot{x}$$

General solution

$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3$$

$$x(t) = C_2\cos(\omega t) - C_1\sin(\omega t) + C_4$$

using initial conditions $\dot{y}(0) = \dot{x}(0) = 0 \qquad y(0) = 0 \qquad x(0) = X_0$

one can find

$$y(t) = -\frac{E_0}{B_0\omega}\sin(\omega t) + \frac{E_0}{B_0}t = \frac{E_0}{B_0\omega}(\omega t - \sin(\omega t))$$

$$x(t) = -\frac{E_0}{B_0\omega}\cos(\omega t) + \frac{E_0}{B_0\omega} = +\frac{E_0}{B_0\omega}(1 - \cos(\omega t))$$

this can be rewritten using $R = \frac{E}{B\omega}$ as

$$(y - R\omega t)^2 + (x - R)^2 = R^2 \qquad \text{origin shifted to } (x_0, 0)$$

this is the equation of a cycloid: