Spring 1999 #6

Liquid Helium boils at $T_0 = 4.2\,K$ when its vapor pressure $P_0 = 1$
The latent heat of vaporization is about $L = 90\ J/mole$. A few liters of the liquid is held in an isulating dewar, with heat influx of $dQ/dt = 1$ watt. A mechanical pump is used to lower the temperature below T_0. The pump can remove a volume of gas $\frac{dV}{dT} = 200$ liter/sec at room temperature, independent of the gas pressure.

a) Find the vapor pressure of the liquid helium at temperature T.

Start with the Clausius-Clapeyron equation:

$$\boxed{\frac{dp}{dT} = \frac{L}{T\Delta v}}$$

Temp

Reif eq 8.5.12

$$\frac{dp}{dT} = \frac{L}{T(v_{gas} - v_{liquid})} \approx \frac{L}{T\, v_{gas}}$$

now assume the gas obeys the ideal gas equation of state:

$$\frac{V_{gas}}{mole} = \frac{RT}{p}$$

$$\frac{dp}{dT} = \left(L\ \frac{joule}{mole}\right)\left(\frac{1}{T\ Kelvin}\right)\left(p\ \frac{Newton}{meter^2}\right)\left(\frac{1}{R\left(\frac{joule}{mole\ \cancel{K}}\right)T(\cancel{K})}\right)$$

per mole

$$\frac{dp}{dT} = \frac{pL}{RT^2}$$

$$\frac{dp}{p} = \frac{L}{R}\left(\frac{dT}{T^2}\right)$$

$$\int_{p_0}^{p} \frac{1}{p'}\,dp' = \frac{L}{R}\int_{T_0}^{T} T'^{-2}\,dT'$$

← T prime

$$\ln(p')\Big]_{p_0}^{p} = \frac{L}{R}\left[-T'^{-1}\right]_{T_0}^{T}$$

⟹

$$\ln\left(\frac{p}{p_0}\right) = \frac{L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)$$

$$\boxed{p = p_0 \exp\left[\frac{L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]}$$

b) What is the minimum temperature of the liquid achieved by the pump?

→ Invert the result from (a) to find T as function of p.

$$\frac{L}{R}\frac{1}{T} = \frac{L}{RT_0} - \ln\left(\frac{p}{p_0}\right)$$

$$\frac{1}{T} = \frac{1}{T_0} - \frac{R}{L}\ln\left(p/p_0\right)$$

$$T = \left[\frac{1}{T_0} - \frac{R}{L}\ln\left(p/p_0\right)\right]^{-1}$$

Now, one only needs to find p, the steady state vapor pressure of the gas. To find p, note that (Q/L) $\frac{\text{mole}}{\text{sec}}$ evaporate and are removed by the pump, so the equation of state is

$$\frac{Q}{L}\left(\frac{\text{mole}}{\text{sec}}\right) = \frac{p}{RT_{300}}\frac{dV}{dt}\left(\frac{\text{mole}}{\text{sec}}\right) \quad \Rightarrow \quad p = \left(\frac{QRT_{300}}{L}\right)\left(1/(dV/dt)\right)$$

plugging in values yields: $p = \frac{(1)(8.3)(300)}{(90)(200\times10^{-3})} = 138\ N/m^2$

(1 liter $= 10^{-3}$

$p_0 = 1.01\times10^5$ Pascal $= 1.01\times10^5\ N/m^2$

$$\boxed{T = \left[\frac{1}{4.2} - \frac{8.3}{90}\ln\left(\frac{138}{10^5}\right)\right]^{-1} K}$$

(using a calculator gives $T = 1.18\ K$)