S'99 #6, Stat. Mech.

Liquid helium boils at a temperature $T_0 = 4.2$ K when its vapor pressure is $P_0 = 1$ atm. The latent heat of vaporization is about $\ell = 90$ joules/mole and can be taken to be temperature independent. A few liters of the liquid is held in an insulating dewar, but with a heat influx to the liquid of $dQ/dt = 1$ Watt. A mechanical pump is used to lower the temperature of the liquid below T_0. The pump can remove a volume of gas $dV/dt = 200$ liters/s at room temperature (300K), independent of the gas pressure.

a) Find the vapor pressure of the liquid helium at temperature T, in terms of the latent heat, P_0, T_0, and numerical and physical constants.

b) What is the minimum temperature of the liquid achieved by the pump?

See Section 8.5 and Problem 8.5 in Reif

$$\frac{dP}{dT} = \frac{\ell}{T\Delta v} = \frac{L}{T\Delta V}$$

(ℓ: molar latent heat; L: latent heat; Δv: molar volume; ΔV: volume)

and apply the following assumptions:

- $\Delta V = V_{gas} - V_{liquid} \approx V_{gas}$
- Treat the gas as an ideal gas (i.e. $PV = nRT$)

so $V = \frac{nRT}{P} \Rightarrow \frac{dP}{dT} = \frac{L}{nRT^2} P = \frac{\ell}{RT^2} P$

so $\frac{dP}{P} = \frac{\ell}{R}\frac{dT}{T^2} \Rightarrow \ln P \Big|_{P_0}^{P} = \frac{\ell}{R}\left(\frac{1}{T}\right)_{T_0}^{T}$

$\Rightarrow \ln(P/P_0) = \frac{\ell}{R}\left[\frac{1}{T_0} - \frac{1}{T}\right] \Rightarrow \boxed{P = P_0 \exp\left[\frac{\ell}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]}$

b) Invert the previous result:

$$\frac{R}{\ell}\ln(P/P_0) = \frac{1}{T_0} - \frac{1}{T} \Rightarrow \frac{1}{T} = \frac{1}{T_0} - \frac{R}{\ell}\ln(P/P_0)$$

so $T(P) = \left[\frac{1}{T_0} - \frac{R}{\ell}\ln(P/P_0)\right]^{-1}$ $T_0 = 4.2\,K$; $P_0 = 1\,atm = 1.01\cdot 10^5\,N/m^2$

$R = 8.31\ J/mol\,K$

need an expression for P:

(assuming that the Latent heat is all the energy needed to vaporize liquid - no additional to bring it to the boiling point)

$\frac{Q}{\ell} = n = \frac{P}{RT}\frac{dV}{dt} \Rightarrow P = \frac{QRT}{\ell}\left(1/\frac{dV}{dt}\right)$; $Q = 1\ Joule/sec$

$T = 300\,K$

$\ell = 90\ Joule/mol$

$\frac{dV}{dt} = 200\,L/s \quad (1L = 10^{-3}\,m^3)$

$= 0.2\ m^3/s$

so $P = 138\ N/m^2$

hence $T = \left[\frac{1}{4.2} - \frac{8.31}{90}\ln\left(\frac{138}{1.01\cdot 10^5}\right)\right]^{-1} = \boxed{1.18\,K}$