2 Scaling

In most of the previous examples, we used opportunistic tricks to determine what numbers to multiply together. We now introduce a new method, **scaling**, for problems where simple multiplication is not sufficient. Instead of explaining what a scaling argument is, we first make one, and then explain what we did. The fastest way to learn a language is to hear and speak it. Physics is no exception; you hear it in the examples, and you speak it in the exercises.

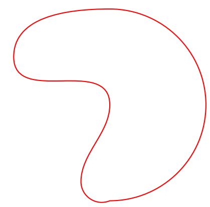


Figure 2.1. *Strange shape with area A.*

2.1 Geometry

The shape in Figure 2.1 has an area A. If all its dimensions are increased by 1.5 to produce Figure 2.2, what is the new area? Let's first do it the silly way. We lay a grid under each object and count squares, finding 15 squares for the small shape (Figure 2.3) and 36 for the large shape (Figure 2.4). With $A = 15$, the new area is

$$A_{\text{new}} = \frac{36}{15} A \approx 2.4 \times A. \tag{2.1}$$

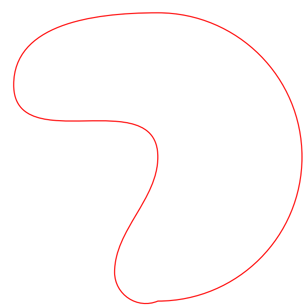


Figure 2.2. *Same shape as Figure 2.1 but with dimensions enlarged by 1.5. What is its area?*

That's the difficult method. Another method is to scale the area. Every length increased by a factor of 1.5, and $A \propto l^2$, so the area increased by a factor of $1.5^2 = 2.25$. This result is close to the estimate of 2.4 in (2.1), or as close as one could expect by counting squares. Furthermore, in spite of being derived from a proportionality, an allegedly approximate relation (where's the equals sign?), the factor of 2.25 is exact. As a bonus, this scaling method is also easier than counting squares. We introduce the idea in this geometry example, where it is so obviously painful, or painfully obvious, that one should compare the new quantity to a known one rather than working out it out from scratch. Now for a physics example of the same moral.

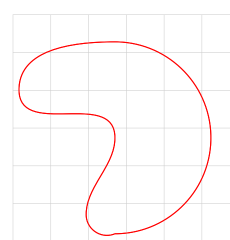


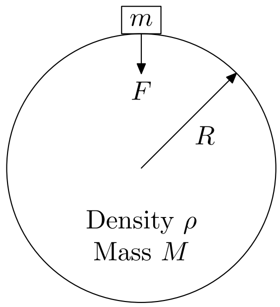
Figure 2.3. *Figure 2.1 with a grid underneath. Counting squares gives an area of roughly 15 grid squares.*

2.2 Gravity on the moon

What is acceleration due to gravity on the surface of the moon?

First, we guess. Should it be $1\,\text{cm}\,\text{s}^{-2}$, or $10^6\,\text{cm}\,\text{s}^{-2}$, or perhaps $10^3\,\text{cm}\,\text{s}^{-2}$? They all sound reasonable, so we make the guess of least resistance—that everywhere is like our local environment—and say that $g_{\text{moon}} \sim g_{\text{earth}}$, which is $1000\,\text{cm}\,\text{s}^{-2}$. Now we will make a systematic estimate.

This method that we use eventually shows you how to make estimates without knowing physical constants, such as the gravitational



$$M \sim \rho R^3$$
$$F = \frac{GMm}{R^2} \propto \rho R m$$

Figure 2.5. *Order-of-magnitude astronomical body. The body, which we approximate as a sphere, has radius R and uniform density ρ. A block of mass m sits on the surface and feels a gravitational force $F = GMm/R^2$, where $M \sim \rho R^3$ is the mass of the astronomical body. The resulting acceleration is $g = F/m = G\rho R \propto \rho R$; if ρ is the same for all astronomical bodies in which we're interested, then $g \propto R$.*

constant G. First, we give the wrong solution, so that we can contrast it with the right—and simpler—order-of-magnitude solution. The acceleration due to gravity at the surface of the moon is given by Newton's law of gravitation (Figure 2.5):

$$g = \frac{F}{m} = \frac{GM}{R^2}. \tag{2.2}$$

In the wrong way, we look up—perhaps in the thorough and useful *CRC Handbook of Chemistry and Physics* [38]—M and R for the moon, and the fundamental constant G, and get

$$g_{\text{moon}} \sim \frac{6.7 \cdot 10^{-8}\,\text{cm}^3\,\text{g}^{-1}\,\text{sec}^{-2} \times 7.3 \cdot 10^{25}\,\text{g}}{(1.7 \cdot 10^8\,\text{cm})^2} \sim 160\,\text{cm}\,\text{s}^{-2}. \tag{2.3}$$

Here is another arithmetic calculation that you can do mentally, perhaps saying to yourself, "First, I count the powers of 10: There are 17 $(-8+25)$ powers of 10 in the numerator, and 16 $(8+8)$ in the denominator, leaving 1 power of 10 after the division. Then, I account for the prefactors, ignoring the factors of 10. The numerator contains 6.7×7.3, which is roughly $7 \times 7 = 49$. The denominator contains $1.7^2 \sim 3$. Therefore, the prefactors produce $49/3 \sim 16$. When we include one power of 10, we get 160."

This brute-force method—looking up every quantity and then doing arithmetic—is easy to understand, and is a reasonable way to get an initial solution. However, it is not instructive. For example, when you compare $g_{\text{moon}} \sim 160\,\text{cm}\,\text{s}^{-1}$ with g_{earth}, you may notice that g_{moon} is smaller than g_{earth} by a factor of only ~ 6. With the huge numbers that we multiplied and divided in (2.3), g_{moon} could easily have been $0.01\,\text{cm}\,\text{s}^{-2}$ or $10^6\,\text{cm}\,\text{s}^{-2}$. Why are g_{moon} and g_{earth} nearly the same, different by a mere factor of 6? The brute-force method shows only that huge numbers among G, M, and R^2 nearly canceled out to produce the moderate acceleration $160\,\text{cm}\,\text{s}^{-2}$.

So we try a more insightful method, which has the benefit that we do not have to know G; we have to know only g_{earth}. This method is not as accurate as the brute-force method, but it will teach us more physics. It is an example of how approximate answers can be more useful than exact answers.

We find g_{moon} for the moon by scaling it against g_{earth}. [It is worth memorizing g_{earth}, because so many of our estimations depend on its

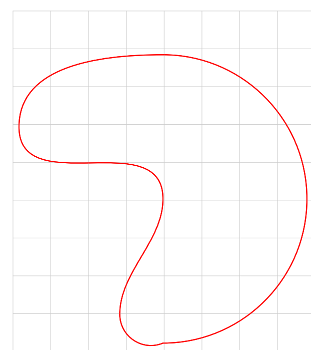


Figure 2.4. *Figure 2.2 with a grid underneath. Counting squares gives an area of roughly 36 grid squares.*

value.] We begin with (2.2). Instead of M and R, we use density ρ and radius R as the independent variables; we lose no information, because we can compute density from mass and radius (assuming, as usual, that the astronomical body has the simplest shape: a sphere). We prefer density to mass, because density and radius are more orthogonal than mass and radius. In a thought experiment—and order-of-magnitude analyses are largely thought experiments—we might imagine a larger moon made out of the same type of rock. Enlarging the moon changes both M and R, but leaves ρ alone. To keep M fixed while changing R requires a larger feat of imagination (we shatter the moon and use scaffolding to hold the fragments at the right distance apart).

For a sphere of constant density, $M = (4\pi/3)\rho R^3$, so (2.2) becomes

$$g \propto \rho R. \tag{2.4}$$

This **scaling relation** tells us how g varies—scales—with density and radius. We retain only those variables and factors that change from the earth to the moon; the proportionality sign $\propto$ allows us to eliminate constants such as G, and numerical factors such as $4\pi/3$.

If the earth and moon have the same radius and the same average density of rock, then we can further simplify (2.4) by eliminating ρ and R to get $g \propto 1$. These assumptions are not accurate, but they simplify the scaling relation; we correct them shortly. So, in this simple model, g_{moon} and g_{earth} are equal, which partially explains the modest factor of 6 that separates g_{moon} and g_{earth}. Now that we roughly understand the factor of 6, as a constant near unity, we strive for more accuracy, and remove the most inaccurate approximations. The first approximation to correct is the assumption that the earth and moon have the same radius. If R can be different on the earth and moon, then (2.4) becomes $g \propto R$, whereupon $g_{\mathrm{earth}}/g_{\mathrm{moon}} \sim R_{\mathrm{earth}}/R_{\mathrm{moon}}$.

What is R_{moon}? Once again, we apply the guerrilla method. When the moon is full, a thumb held at arms length will just cover the moon perceived by a human eye. For a typical human-arm length of 100 cm, and a typical thumb width of 1 cm, the angle subtended is $\theta \sim 0.01$ rad. The moon is $L \sim 4 \cdot 10^{10}$ cm from the earth, so its diameter is $\theta L \sim 0.01L$; therefore, $R_{\mathrm{moon}} \sim 2 \cdot 10^8$ cm. By contrast, $R_{\mathrm{earth}} \sim 6 \cdot 10^8$ cm, so $g_{\mathrm{earth}}/g_{\mathrm{moon}} \sim 3$. We have already explained a large part of the factor of 6. Before we explain the remainder, let's estimate L from familiar parameters of the moon's orbit. One of the goals of order-of-magnitude physics is to show you that you can make many estimates with the knowledge that you already have. Let's apply this philosophy to estimating the radius of the moon's orbit. One familiar parameter is the period: $T \sim 30$ days. The moon orbits in a circle because of the earth's gravitational field. What is the effect of earth's gravity at distance L (from the center of the earth)? At distance R_{earth} from the center of the earth, the acceleration due to gravity is g; at L, it is $a = g(R_{\text{earth}}/L)^2$, because gravitational force (and, therefore, acceleration) are proportional to distance^{-2}. The acceleration required to move the moon in a circle is v^2/L. In terms of the period, which we know, this acceleration is $a = (2\pi L/T)^2/L$. So

$$\underbrace{g\left(\frac{R_{\text{earth}}}{L}\right)^2}_{a_{\text{gravity}}} = \underbrace{\left(\frac{2\pi L}{T}\right)^2 \frac{1}{L}}_{a_{\text{required}}}. \tag{2.5}$$

The orbit radius is

$$\begin{aligned} L &= \left(\frac{gR_{\text{earth}}^2 T^2}{4\pi^2}\right)^{1/3} \\ &\sim \left(\frac{1000\,\text{cm}\,\text{s}^{-2} \times (6\cdot 10^8\,\text{cm})^2 \times (3\cdot 10^6\,\text{sec})^2}{40}\right)^{1/3} \\ &\sim 5\cdot 10^{10}\,\text{cm}, \end{aligned} \tag{2.6}$$

which closely matches the actual value of $4\cdot 10^{10}$ cm.

Now we return to explaining the factor of 6. We have already explained a factor of 3. (A factor of 3 is more than one-half of a factor of 6. Why?) The remaining error (a factor of 2) must come largely because we assumed that the earth and moon have the same density. Allowing the density to vary, we recover the original scaling relation (2.4). Then,

$$\frac{g_{\text{earth}}}{g_{\text{moon}}} \sim \frac{\rho_{\text{earth}}}{\rho_{\text{moon}}} \frac{R_{\text{earth}}}{R_{\text{moon}}}. \tag{2.7}$$

Typically, $\rho_{\text{crust}} \sim \rho_{\text{moon}} \sim 3\,\text{g}\,\text{cm}^{-3}$, whereas $\rho_{\text{earth}} \sim 5\,\text{g}\,\text{cm}^{-3}$ (here, ρ_{crust} is the density of the earth's crust).

Although we did not show you how to deduce the density of moon rock from well-known numbers, we repay the debt by presenting a speculation that results from comparing the average densities of the earth and the moon. Moon rock is lighter than earth rock; rocks in the earth's crust are also lighter than the average earth rock (here "rock" is used to include all materials that make up the earth, including the core, which is nickel and iron); when the earth was young, the heavier, and therefore denser, elements sank to the center of the earth. In fact, moon rock has density close to that of the earth's crust—perhaps because the moon was carved out of the earth's crust. Even if this hypothesis is not true, it is plausible, and it suggests experiments that might disprove it. Its genesis shows an advantage of the scaling method over the brute-force method: The scaling method forces us to compare the properties of one system with the properties of another. In making that comparison, we may find an interesting hypothesis.

Whatever the early history of the moon, the density ratio contributes a factor of 5/3 or roughly 1.7 to the ratio (2.7), and we get

$g_{\rm earth}/g_{\rm moon} \sim 3 \times 1.7 \sim 5$. We have explained most of the factor of 6—as much of it as we can expect, given the crude method that we used to estimate the moon's radius, and the one-digit accuracy that we used for the densities.

The brute-force method—looking up all the relevant numbers in a table—defeats the purpose of order-of-magnitude analysis. Instead of approximating, you use precise values and get a precise answer. You combine numerous physical effects into one equation, so you cannot easily discern which effects are important. The scaling method, where we first approximate the earth and moon as having the same density and radius, and then correct the most inaccurate assumptions, teaches us more. It explains why $g_{\rm moon} \sim g_{\rm earth}$: because the earth and moon are made of similar material and are roughly the same size. It explains why $g_{\rm moon}/g_{\rm earth} \simeq 1/6$: because moon rock is lighter than earth rock, and because the moon is smaller than the earth. We found a series of successive approximations:

$$\begin{aligned} g_{\rm moon} &\sim g_{\rm earth}, \\ g_{\rm moon} &\sim \frac{R_{\rm moon}}{R_{\rm earth}}\, g_{\rm earth}, \\ g_{\rm moon} &\sim \frac{\rho_{\rm moon}}{\rho_{\rm earth}} \frac{R_{\rm moon}}{R_{\rm earth}}\, g_{\rm earth}. \end{aligned} \tag{2.8}$$

Since the approximations each introduce only one physical effect, they are easy to understand. Another benefit of the scaling method is that it can suggest new theories or hypotheses. When we considered the density of moon rock and earth rock, we were led to speculate on the moon's origin from the earth's crust. Order-of-magnitude reasoning highlights the important factors, so that our limited brains can digest them, draw conclusions from them, and possibly extend them.

2.3 Collisions

Imagine that you work for a government safety agency testing how safe various cars are in crashes. Your budget is slim, so you first crash small toy cars, not real cars, into brick walls. (Actually, you might crash cars in computer simulation only, but, as the order-of-magnitude analysis of computer programs is not the topic of this example, we ignore this possibility.) At what speed does such a crash produce mangled and twisted metal? Metal toy cars are still available (although hard to find), and we assume that you are using them.

For an initial guess, let's estimate that the speed should be 50 mph or 80 kph—roughly the same speed that would badly mangle a real car (mangle the panels and the engine compartment, not just the fenders). Why does a crash make metal bend? Because the kinetic energy from the crash distorts the metallic bonds. We determine the

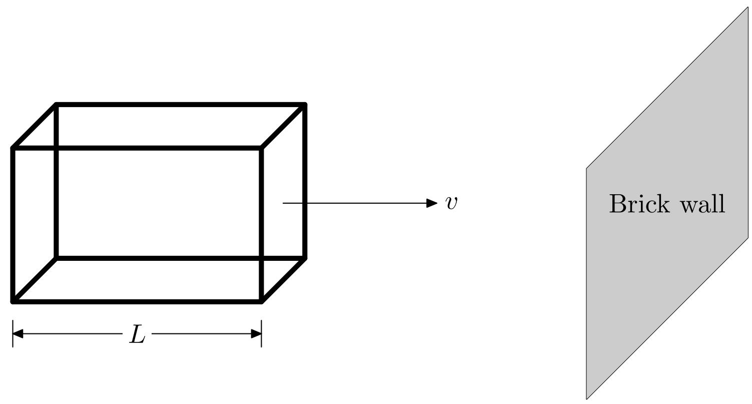


Figure 2.6. *Order-of-magnitude car about to hit a brick wall. It hits with speed v, which provides kinetic energy $\sim Mv^2$, where M is the mass of the car. The energy required to distort a fixed fraction of the bonds is proportional to the number of bonds. If toy and real cars are made of the same metal, then the number of atoms, and the total bond-distortion energy, will be proportional to M, the mass of the car. The available kinetic energy also is proportional to M, so the necessary crash velocity is the same at all masses, and, therefore, at all sizes.*

necessary crash speed using a scaling argument.

Figure 2.6 shows a car about to hit a brick wall. In an order-of-magnitude world, all cars, toy or real, have the same proportions, so the only variable that distinguishes them is their length, L. (Because we are assuming that all cars have the same proportions, we could use the width or height instead of the length.) The kinetic energy available is

$$E_{\text{kinetic}} \sim Mv^2. \tag{2.9}$$

The energy required to distort the bonds is

$$E_{\text{required}} \sim \underbrace{\frac{M}{m_{\text{atom}}}}_{\textit{no. of atoms}} \times \epsilon_c \times f, \tag{2.10}$$

where ϵ_c is the binding, or cohesive, energy per atom; and f is a fractional fudge factor thrown in because the crash does not need to break every bond. We discuss and estimate cohesive energies in Section 4.2.2; for now, we need to know only that the cohesive energy is an estimate of how strong the bonds in the substance are. Let's assume that, to mangle metal, the collision must break a fixed fraction of the bonds, perhaps $f \sim 0.01$. Equating the available energy (2.9) and the required energy (2.10), we find that

$$Mv^2 \sim M \times \frac{\epsilon_c}{m_{\text{atom}}} \times f. \tag{2.11}$$

We assume (reasonably) that ϵ_c, f, and m_{atom} are the same for all cars, toy or real, so once we cancel M, we have $v \propto 1$. The required speed is the same at all sizes, as we had guessed.

Now that we have a zeroth-order understanding of the problem, we can improve our analysis, which assumed that all cars have the same shape. The metal in toy cars is proportionally thicker than the metal in real cars, just as roads on maps are proportionally wider than real roads. So a toy car has a larger mass, and is therefore stronger than the simple scaling predicts. The metal in full-size cars mangles in a 80 kph crash; the metal in toy cars may survive an 80 kph crash, and may mangle only at a significantly higher speed, such as 200 kph.

Our solution shows the benefit of optimism. We do not know the fudge factor f, or the cohesive energy ϵ_c, but if we assume that they are the same for all cars, toy or real, then we can ignore them. The moral is this: **Use symbols** for quantities that you do not know; they might cancel at the end. Our example illustrated another technique: successive approximation. We made a reasonable analysis—implicitly assuming that all cars have the same shape—then improved it. The initial analysis was simple, and the correction was almost as simple. Doing the more accurate analysis in one step would have been more difficult.

2.4 Jump heights

We next apply scaling methods to understand how high an animal can jump, as a function of its size. We study a jump from standing (or from rest, for animals that do not stand); a running jump depends on different physics. This jump-height problem also looks underspecified. The height depends on how much muscle an animal has, how efficient the muscles are, what the animal's shape is, and much else. So we invoke another order-of-magnitude method: When the going gets tough, **lower your standards**. We cannot easily figure out the absolute height; we estimate instead how the height depends on size, leaving the constant of proportionality to be determined by experiment. First we develop a simple model of jumping; then in Section 2.5 we consider physical effects that we neglected in the crude approximations.

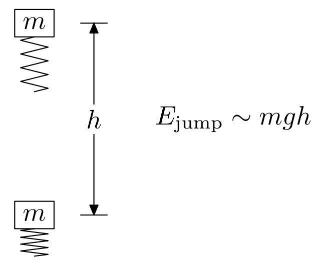


Figure 2.7. *Jumping animal. An animal of mass m (the block) stores energy in its muscles (the compressed, massless spring). It uses the energy to jump a height h off the ground. The energy required is $E_{\text{jump}} \sim mgh$.*

We want to determine only how jump height scales (varies) with body mass. Even this problem looks difficult; the height still depends on muscle efficiency, and so on. Let's see how far we get by just plowing along, and using symbols for the unknown quantities. Maybe all the unknowns cancel. We want an equation for the height h, such as $h \sim f(m)$, where m is the animal's mass. Jumping requires energy, which must be provided by muscles. [Muscles get their energy from sugar, which gets its energy from sunlight, but we are not concerned with the ultimate origins of energy here.] If we can determine the required energy, and compare it with the energy that all the muscles in an animal can supply, then we have an equation for f. Figure 2.7 shows a cartoon version of the problem.

A jump of height h requires energy $E_{\text{jump}} \sim mgh$. So we can write

$$E_{\text{jump}} \propto mh. \tag{2.12}$$

The $\propto$ sign means that we do not have to worry about making the units on both sides match. We exploited this freedom to get rid of the irrelevant constant g (which is the same for all animals on the earth, unless some animal has discovered antigravity). The energy that the animal can produce depends on many factors. We use symbols for each

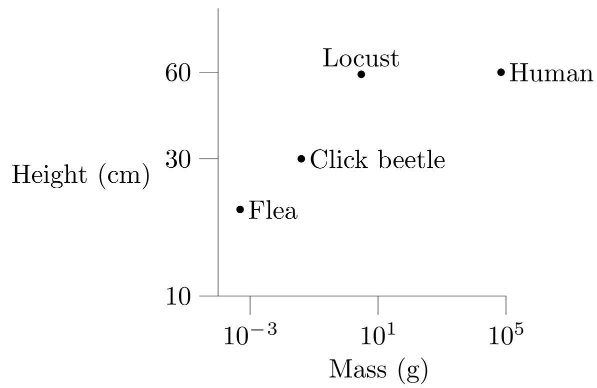


Figure 2.8. *Jump height versus body mass. This graph plots the data in Table 2.1. Notice the small range of variation in height, compared to the range of variations in mass. The mass varies more than 8 orders of magnitude (a factor of 10^8), yet the jump height varies only by a factor of 3. The predicted scaling of constant h ($h \propto 1$) is surprisingly accurate. The largest error shows up at the light end; fleas and beetles do not jump as high as larger animals, due to air resistance.*

of these unknowns. First, the energy depends on how much muscle an animal has. So we approximate by assuming that a fraction, α, of an animal's mass is muscle, and that all muscle tissue can store the same energy density, $\mathcal{E}$ (we are optimists). Then, the energy that can be stored in muscles is

$$E_{\text{stored}} \sim m\alpha\mathcal{E} \propto m. \tag{2.13}$$

Here we have derived a scaling relation, showing how energy stored varies with mass; we used the freedom provided by $\propto$ to get rid of α and $\mathcal{E}$, presumed to be the same for all animals. Equating the required energy from (2.12) with the available energy from (2.13), we find that $mh \propto m$, or that $h \propto 1$; this proportionality says that h is independent of mass. This result seems surprising. Our intuition tells us that people should be able to jump higher than locusts. Table 2.1 shows measured jump heights for animals of various sizes and shapes; the data are also plotted in Figure 2.8. Surprising or not, our result is roughly correct.

Animal	*Mass* (g)	*Height* (cm)
Flea	$0.5 \cdot 10^{-3}$	20
Click beetle	0.04	30
Locust	3	59
Human	$7 \cdot 10^4$	60

Table 2.1. *Jump height as a function of mass. Source:* Scaling: Why Animal Size is So Important *[55, page 178].*

2.5 Jump heights refined

Now that we have a crude understanding of the situation—that jump height is constant—we try to explain more subtle effects. For example, the scaling breaks down for tiny animals such as fleas; they do not jump as high as we expect. What could limit the jump heights for tiny animals? Smaller animals have a larger surface-to-volume ratio than do large animals, so any effect that depends on the surface area is more important for a small animal. One such effect is air resistance; the drag force F on an animal of size L is $F \propto L^2$, as we show in Section 3.4.3. The resulting deceleration is $F/m \propto L^{-1}$, so small animals (small L) get decelerated more than big animals. We would have to include the constants of proportionality to check whether the effect is sufficiently large to make a difference; for example, it could be a negligible effect for large animals, and 10 times as large for small animals, but still be negligible. If we made the estimate, we would find that the effect of air resistance is important, and can partially explain why fleas do not jump as high as humans. The constant jump height also fails for large animals such as elephants, who would break their bones when they landed if they jumped as high as humans.

You might object that treating muscle simply as an energy storage medium ignores a lot of physics. This criticism is valid, but if the basic model is correct, it's simpler to improve the model one step at a time, instead of including every effect in the early stages. As an example of successive refinement, let's consider the power requirements for jumping. How does the power required scale with animal size, and do limitations on power prevent animals from attaining their theoretical jump height?

Power is energy per time; in this case, it is energy required for the jump divided by time during which the energy is released. In (2.12) we found that $E \propto mh$; because h is constant, $E \propto m$. [Successive refinement, which we are doing here, depends on an at least rudimentary understanding of the problem. If we had not already solved the problem crudely, we would not know that $E \propto m$ or that $h \propto 1$.]

We now need to estimate the time required to release the energy, which is roughly the time during which the animal touches the ground while launching. Suppose that the animal blasts off with velocity v. The animal squats to zero height, the clock starts ticking, and the animals starts to push. At the end of the push, when the clock stops ticking, the animal is moving with speed v; we assume that it moves with the same speed throughout its launch (the rectangle assumption). The clock, therefore, stops ticking at time $\tau \sim L/v$. The takeoff speed v is roughly the same for all animals, because $v \propto \sqrt{gh} \propto \sqrt{h}$, and h is roughly constant. So $\tau \propto L$.

How does the energy vary with L? We make the simplest assumption—that all animals have the same density and the same cubical shape. Then, $E \propto m$, and $m \propto L^3$, so $E \propto L^3$.

From our estimates for the energy and the time, we estimate that the power required is $P \sim E/\tau \propto L^2$. Per unit volume, the power required is $\mathcal{P}_{\text{req}} \sim L^{-1}$. If there is a maximum power per unit volume, $\mathcal{P}_{\text{max}}$, that an animal can generate, then sufficiently tiny animals—for whom $\mathcal{P}_{\text{req}}$ is large—might not be able to generate sufficient power. Click beetles overcome this problem by storing energy in their exoskeleton, and jumping only after they have stored sufficient energy: They increase the effective τ, and thus decrease $\mathcal{P}_{\text{req}}$.

The analysis of this extreme case—tiny animals—and the analysis of the power requirements show the value of making a simple analysis, and then refining it. To complete the more detailed analysis, we required results from the simple analysis. If we had tried to include all factors—such as air resistance, bone breakage, power consumption, and energy storage—from the beginning, we would have cooked up a conceptual goulash, and would have had trouble digesting the mess.

The approximate model provides a structure on which we can build the more detailed analyses.

2.6 What you have learned

- Avoid computing a quantity from scratch. Rather, compare it to a previously computed quantity.
- Use proportionality relations. They allow you to ignore constants that remain the same in two situations, so that the constants do not clutter our thinking.
- Imagine scaling a physical system up or down in size and consider how the relevant parameters (area, volume, heat flow, power, etc.) vary with size.

2.7 Exercises

► **2.7** *Moment of inertia*
How does moment of inertia scale with length (keeping density constant)?

► **2.8** *Spheres*
If you double the radius of a 8-dimensional sphere (for comparison, the earth is a 3-dimensional sphere), what happens to its surface 'area'?

► **2.9** *Mars year*
How long is a 'year' on Mars (distance from sun $\sim 2.310^{11}$ m)?

► **2.10** *Range*
Imagine throwing a rock or kicking a ball, and neglect air resistance. Keep the launch angle constant. How does the time in the air scale with launch velocity v? How does time in the air scale with g, the gravitational acceleration? Combine these two results to find how the range scales with v and g and thereby deduce a dimensionally correct formula (i.e. with the same dimensions on both sides, even if the formula lacks a constant).

► **2.11** *Bugs*
Surface tension is force per length. Show that a small enough bug (perhaps smaller than any existing bug!) can float on water.

► **2.12** *Mountains*
How does the maximum height of a (cubical!) mountain scale with the radius of a planet?