5 Mechanical properties

Why are mountains on earth at most 10 km high (Section 5.9)? Why must a Thanksgiving turkey cook for many hours (Example 6.2)? Why are densities of most liquids and solids in the range 1–10 g cm^{-3} (Section 5.4)? How large are raindrops (Section 5.7)? How large are white dwarfs (Section 22.1)? Why does water boil at 100 °C (Section 6.3)? Mountains, turkeys, densities, drops, white dwarfs, and water: Their physics depends on the mechanical and thermal properties of materials. Mechanical properties are the subject of this chapter. Thermal properties, which depend on several of the mechanical properties, are the subject of Chapter 6. The techniques of the previous chapters, along with two new techniques – balancing and order-of-magnitude differentiation – make the analyses and estimates tractable and, we hope, enjoyable.

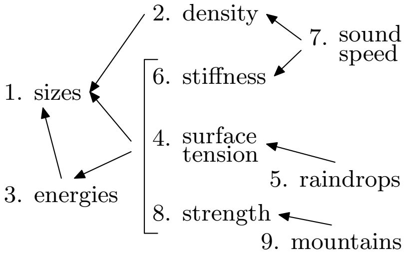

Figure 5.1. Concept map. An arrow from concept A to concept B means that understanding A depends on understanding B. The concepts grouped by the bracket require understanding both sizes and energies. The numbering indicates one possible order in which no concept is studied before its prerequisites. Other orders also satisfy this dependency requirement: for example stiffness and sound speed before surface tension and raindrops.

All material properties depend on the **atomic theory**. In Feynman's words [14, p. 1-2]:

> If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the *atomic hypothesis* (or the atomic *fact*, or whatever you wish to call it) that *all things are made of atoms – little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.* In that one sentence, you will see, there is an *enormous* amount of information about the world...

The atomic theory was first stated by Democritus. (Early Greek science and philosophy is discussed with wit, sympathy, and insight in Bertrand Russell's *History of Western Philosophy* [55].) Democritus could not say much about the properties of atoms. With modern knowledge of classical and quantum mechanics, you can say more.

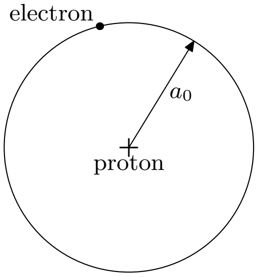

Figure 5.2. Hydrogen. A proton (the "+") holds an electron in jail. The jail radius is a_0, the **Bohr radius.**

This chapter studies atomic (and molecular) sizes and energies and then their consequences: density, surface tension, stiffness, sound speed, and strength. Density depends only on sizes, whereas – as we show in this chapter – stiffness, surface tension, and strength depend on sizes and energies; and sound speed depends on stiffness and density. These relationships, shown in Figure 5.1, suggest a possible order of analysis, which the chapter follows: sizes, densities, energies, surface tension, stiffness, sound speed, and then strength.

5.1 Sizes by dimensional analysis

All matter is made of atoms, so study first the size of atoms, Always draw a simple picture, here of the simplest atom, hydrogen (Figure 5.2). What is the radius of the orbit? Facing confusion, try dimensional analysis. The first step is to list the relevant variables by considering what physics determines the properties of hydrogen. The electrostatic force holds the electron in orbit, so e, the charge on the electron, is on the list. Use e^2 rather than e because, in cgs units,

$$\frac{e^2}{r} \tag{5.1}$$

is a potential energy, so e^2 does not contain dimensions of charge. If you use e, then you face the problem of how to combine charge with familiar dimensions such as length. In SI (mks) units, you would introduce the permittivity of free space, ϵ_0. Then

$$\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \tag{5.2}$$

has dimensions of energy. Symbolic results are often simpler in cgs units, which avoid the ghastly $4\pi\epsilon_0$, than they are in SI units. Numerical estimates are often simpler in SI units because tables quote resistances in ohms rather than statohms, or charges in Coulombs rather than esu. So when you think, use cgs units; when you calculate, use SI units. Get comfortable in both systems!

Var.	*Dim.*	*Description*
a_0	L	radius
e^2	$\mathrm{ML^3T^{-2}}$	charge^2
$\hbar$	$\mathrm{ML^2T^{-1}}$	quantum mechanics
m_p	M	proton mass
m_e	M	electron mass

Table 5.1. *Variables that determine the size of hydrogen. The squared charge is in cgs units, where e^2/r is an energy (without needing a $4\pi\epsilon_0$).*

The mass of the electron m_e determines the acceleration that the electrostatic force produces, so m_e is on the list. The mass of the proton m_p might also be relevant. These four variables – a_0, e^2, m_e, and m_p – containing three independent dimensions produce one dimensionless group (by the Buckingham Pi theorem). Two variables from the list have dimensions of mass, so their ratio is a dimensionless group:

$$\Pi_1 \equiv \frac{m_\mathrm{e}}{m_\mathrm{p}}. \tag{5.3}$$

That group, sadly, does not contain a_0, so you cannot solve for a_0 with this method. The failure to find a useful dimensionless group tells you that the analysis has omitted physics that determines a_0.

As a clue to this missing physics, the picture of hydrogen in Figure 5.2 cannot be classical. A classical electron, moving in a circle, would radiate, lose energy, and spiral into its doom as the proton swallowed it; classical atoms cannot exist. Fortunately, quantum mechanics comes to the rescue. Unfortunately, quantum mechanics is a large, fearsome subject. Thanks to dimensional analysis, you do not have to understand quantum mechanics to compute results based on it. For the purposes of dimensional analysis, the content of quantum mechanics is contained in another constant of nature: $\hbar$, whose value is $1.05\cdot10^{-34}$ J s. Adding $\hbar$ to the list gives the list in Table 5.1.

One more variable produces one more dimensionless group. You are sunk if this group does not contain a_0, so assume that it contains a_0, and assume that it lives upstairs for simplicity. How to make a length from the other four variables, to cancel the length in a_0? Divide and conquer. Consider the dimensions one at a time. Only time shows up in exactly two variables: in e^2 and $\hbar$. So the length being manufactured must contain $\hbar^2/e^2$, or some power of it (including the zeroth power), because the ratio $\hbar^2/e^2$ has no powers of time in it. However, the ratio has one power of mass. Either m_e of m_p can cancel the mass. For now try m_e; we shortly discuss what happens if you choose m_p. So

$$l \equiv \frac{\hbar^2}{m_\text{e}e^2}, \tag{5.4}$$

is a length. The new dimensionless group is

$$\Pi_2 = \frac{a_0}{l} = \frac{a_0}{\hbar^2/m_\text{e}e^2}. \tag{5.5}$$

The other group is still $\Pi_1 = m_\text{e}/m_\text{p}$. Since a_0 lives in the second group, use this form to solve for a_0:

$$\Pi_2 = f(\Pi_1), \tag{5.6}$$

where f is a dimensionless function. Thus

$$a_0 = \frac{\hbar^2}{m_\text{e}e^2}\, f\left(\frac{m_\text{e}}{m_\text{p}}\right). \tag{5.7}$$

How does f behave? The proton is much more massive than the electron, so study the limiting case: pretend that $m_\text{e}/m_\text{p} = 0$ and study the behavior of $f(x)$ as $x \to 0$, where $x \equiv m_\text{e}/m_\text{p}$. If $f(x)$ has a limit as $x \to 0$, then (5.7) becomes

$$a_0 \sim \frac{\hbar^2}{m_\text{e}e^2}, \tag{5.8}$$

where the missing constant is $f(0)$. For atoms with infinitely massive nuclei, i.e. for $m_\text{e}/m_\text{p} \to 0$, the constant $f(0)$ turns out to be 1. The Bohr radius is so important that the calculation deserves a cross-check, which is the subject of Section 5.2.

Meanwhile you might wonder whether the limit $f(0)$ exists. Depending on how you form the groups, it might not. Suppose that you use m_p rather than m_e to construct Π_2:

$$\Pi_2 = \frac{a_0}{\hbar^2/m_\text{p}e^2} \qquad \text{(don't do this)}. \tag{5.9}$$

The solution (5.7) becomes

$$a_0 = \frac{\hbar^2}{m_\text{p}e^2}\, f\left(\frac{m_\text{e}}{m_\text{p}}\right). \tag{5.10}$$

Now do the same thought experiment as after (5.7): imagine taking m_p to ∞. In this limit, the proton moves less and less, so its mass becomes more and more irrelevant to the orbit of the electron. Therefore m_p should vanish from a_0. Its absence requires that $f(x) \sim 1/x$; then the resulting factor of $m_\mathrm{p}/m_\mathrm{e}$ in $f(x)$ cancels the m_p outside (in red). In this analysis, $f(x)$ goes to infinity as $x \to 0$, so $f(0)$ does not exist. The moral: Sloppy physics produces unpleasant mathematics.

How can you decide which mass, m_e or m_p, to include in the group Π_2 in order to get an f that behaves well at 0? You can try both alternatives or try the right one by luck. No recipe is foolproof, but a rule of thumb is to guess which variable is likely to vanish and to isolate it in one dimensionless group. We applied this rule in the drag problem when revising the second dimensionless group (4.11) not to include viscosity. In this example, the disparity between the proton and electron masses (their ratio is ~ 2000) suggests extending the disparity, at least in a thought experiment. You can increase m_p or decrease m_e. By analyzing the consequence of each change, you might learn which mass is the more important. The hydrogen atom, in this simple world, is an isolated system with no net force on it. So the momenta of the proton and the electron are equal and opposite with, say, magnitude P. A typical kinetic energy is $E \sim P^2/m$. The mass is downstairs, so the heavier particle (the proton) contributes much less energy than the lighter particle (the electron) contributes. As m_p increases, only a tiny contributor to the total energy changes. As m_e decreases, however, the total energy changes significantly. This argument suggests that hydrogen has a physical limit when $m_\mathrm{p} \to \infty$ and m_e stays fixed, whereas not in the opposite limit of $m_\mathrm{e} \to 0$ with m_p fixed. So m_p is more likely to be thrown away than m_e is – which means that you should isolate m_p in one dimensionless group.

You could arrive at the opposite conclusion by a similar reasoning that uses $1/E$ instead of E. In $1/E$ mass is upstairs so m_p becomes more important than m_e. Mathematically this argument is possible. However, $1/E$ is a lousy physical quantity. Unlike E, it is not conserved. So reasoning about it is not likely to produce mathematical conclusions that help improve physical understanding. In short, reasonable physics suggests that m_p is less important than m_e.

5.2 Sizes by physics

The previous method, dimensional analysis, is mostly mathematical. As a second computation of a_0, we show you a method that is mostly physics. Besides checking the Bohr radius result (5.8), it provides a physical interpretation of it. The Bohr radius is the radius of the orbit with the lowest energy (the ground state). The energy is a sum of kinetic and potential energy. This division suggests, again, a divide-and-conquer approach: first the kinetic energy, then the potential energy.

What is the origin of the kinetic energy? The electron does not orbit in any classical sense. If it orbited, it would, as an accelerating charge, radiate energy and spiral into the nucleus. According to quantum mechanics, however, the proton confines the electron to a region of size r – still unknown to us – and the electron exists in a so-called stationary state. The nature of a stationary state is mysterious; no one understands quantum mechanics, so no one understands stationary states except mathematically. However, in an order-of-magnitude estimate you can ignore details such as the meaning of a stationary state. The necessary information here is that the electron is, as the name of the state suggests, stationary: It does not radiate. The problem then is to find the size of the region to which the electron is confined. In reality the electron is smeared over the whole universe; however, a significant amount of it lives within a typical radius. This typical radius we estimate and call a_0.

For now let this radius be an unknown r and study how the kinetic energy depends on r. Confinement gives energy to the electron according to the **uncertainty principle**:

$$\Delta x \Delta p \sim \hbar, \tag{5.11}$$

where Δx is the position uncertainty and Δp is the momentum uncertainty of the electron. In this model $\Delta x \sim r$, as shown in Figure 5.3, so $\Delta p \sim \hbar/r$. The kinetic energy of the electron is

$$\mathrm{KE} \sim \frac{(\Delta p)^2}{m_\mathrm{e}} \sim \frac{\hbar^2}{m_\mathrm{e} r^2}. \tag{5.12}$$

This energy is the **confinement energy** or the **uncertainty energy**. This idea recurs in the book.

The potential energy is the classical expression

$$\mathrm{PE} \sim -\frac{e^2}{r}. \tag{5.13}$$

This form is in cgs units, and you can leave it that way until you need to calculate a quantity with e^2 in it. The classical potential energy is exactly $-e^2/r$, but the $\sim$ sign indicates that the electron is not precisely at a radius r. Rather, its typical or characteristic distance from the proton is r.

The total energy is the combination

$$E = \mathrm{PE} + \mathrm{KE} \sim -\frac{e^2}{r} + \frac{\hbar^2}{m_\mathrm{e} r^2}. \tag{5.14}$$

The two energies compete. At small r, kinetic energy wins, because of the $1/r^2$; at large r, potential energy wins, because it goes to zero less

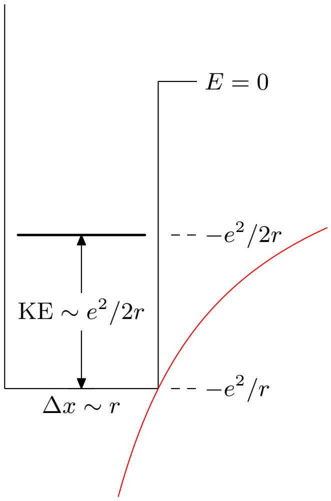

Figure 5.3. *Order-of-magnitude hydrogen. The Coulomb potential (red curve) is approximately a box potential (solid line). Its width and height are not known until you compute r and then the ground-state energy. But you compute r and the ground-state energy by replacing the Coulomb potential, in which quantum mechanics is mathematically complex, with a box potential, where it is easier to compute the ground-state energy. This replacement is the rectangle approximation introduced to estimate the number of babies born every year (Section 1.4). The total energy of the electron (thick line) is the sum of the potential energy, $-e^2/r$, and the kinetic energy, $e^2/2r$.*

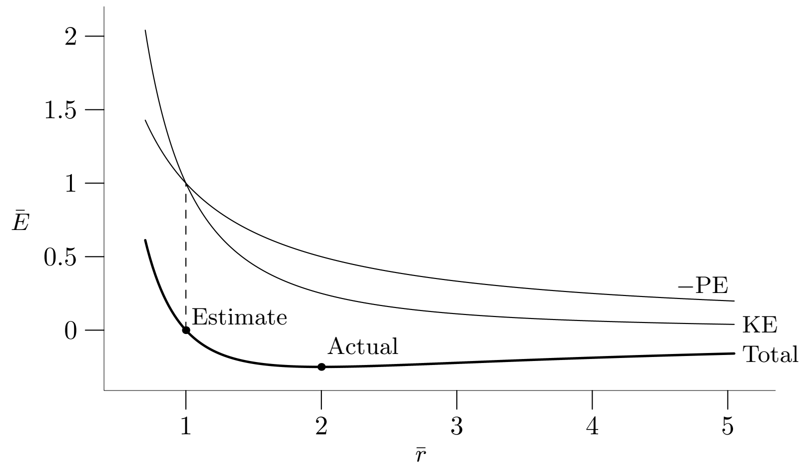

Figure 5.4. *Order-of-magnitude calculus: minimizing scaled energy* $\bar{E}$ *versus scaled bond length* $\bar{r}$*. The scaled energy is the sum of potential and kinetic energy. The shape of this energy illustrates Feynman's explanation of the atomic hypothesis (p. 65). At a 'little distance apart' – for large* $\bar{r}$ *– the curve slopes upward; to lower their energy, the proton and electron prefer to move closer, and the resulting force is attractive. 'Upon being squeezed into one another' – for small* $\bar{r}$ *– the potential rapidly increases, so the force between the particles is repulsive. Somewhere between the small and large regions of* $\bar{r}$*, the force is zero.*

rapidly. Is there a minimum combined energy at some intermediate value of r? There has to be. At small r, the slope dE/dr is negative. At large r, it is positive. At an intermediate r, the slope crosses between positive and negative. The energy is a a minimum there. The location would be easy to estimate if (5.14) were in dimensionless form. Such a rewriting is not necessary in this example, but it is helpful in more complicated examples. In constructing the dimensionless group (5.5) containing a_0, we constructed another length:

$$l = \frac{\hbar^2}{m_e e^2}. \tag{5.15}$$

You can scale any length – make it dimensionless – by dividing it by l. So in the total energy (5.14), use

$$\bar{r} \equiv \frac{r}{l}. \tag{5.16}$$

The other unknown in the total energy is the energy itself. To make it dimensionless, a reasonable energy scale to use is e^2/l. So define scaled energy as

$$\bar{E} \equiv \frac{E}{e^2/l}. \tag{5.17}$$

Using the dimensionless length (5.16) and energy (5.17), the total energy (5.14) becomes

$$\bar{E} \sim -\frac{1}{\bar{r}} + \frac{1}{\bar{r}^2}. \tag{5.18}$$

The ugly constants have vanished; they hide in the definitions of scaled length and energy. This dimensionless energy is easy to think about and to sketch (Figure 5.4).

Calculus (differentiation) locates this minimum-energy $\bar{r}$ at $\bar{r}_{\text{min}} = 2$. An alternative method is **order-of-magnitude minimization**: When two terms compete, the minimum occurs when the terms are roughly equal. Equating the two terms $\bar{r}^{-1}$ and $\bar{r}^{-2}$ gives $\bar{r}_{\text{min}} \sim 1$.

This result is a scaled length. In normal units, it is

$$r_{\text{min}} = l\bar{r}_{\text{min}} = \frac{\hbar^2}{m_{\text{e}}e^2}, \tag{5.19}$$

which is the Bohr radius (5.8) from dimensional analysis. The sloppiness in estimating the kinetic and potential energies has canceled the error introduced by order-of-magnitude minimization!

To justify order-of-magnitude minimization, consider a reasonable general form for E:

$$E(r) = \frac{A}{r^n} - \frac{B}{r^m}. \tag{5.20}$$

This form captures the important feature of (5.14): The two terms represent competing physical effects, represented mathematically by their opposite signs. To find the minimum, solve $E'(r_{\text{min}}) = 0$ or

$$-n\frac{A}{r_{\text{min}}^{n+1}} + m\frac{B}{r_{\text{min}}^{m+1}} = 0. \tag{5.21}$$

The solution is

$$\frac{A}{r_{\text{min}}^n} = \frac{n}{m}\frac{B}{r_{\text{min}}^m} \qquad \text{(calculus)}. \tag{5.22}$$

The order-of-magnitude method minimizes (5.20) by equating the two terms A/r^n and B/r^m:

$$\frac{A}{r_{\text{min}}^n} = \frac{B}{r_{\text{min}}^m} \qquad \text{(oom)}. \tag{5.23}$$

This approximation lacks the n/m factor in (5.22). The ratio of the two estimates for r_{min} is

$$\frac{\text{order-of-magnitude estimate}}{\text{calculus estimate}} \sim \left(\frac{n}{m}\right)^{1/(m-n)}, \tag{5.24}$$

which is smaller than 1 unless $n = m$, when there is no maximum or minimum. So the order-of-magnitude method underestimates the location of minima and maxima.

To judge the method in practice, apply it to a typical example: the potential between nonpolar atoms or molecules, such as between helium, xenon, or methane. This potential is well approximated by the so-called Lennard–Jones potential, which has $m = 6$ and $n = 12$:

$$U(r) \sim \frac{A}{r^{12}} - \frac{B}{r^6}. \tag{5.25}$$

The order-of-magnitude result (5.23), underestimates r_{min} by a factor of

$$\left(\frac{12}{6}\right)^{1/6} \sim 1.15. \tag{5.26}$$

An error of 15 percent is often small compared to the other inaccuracies in an order-of-magnitude computation.

Now return to the original problem: determining the Bohr radius. The order-of-magnitude minimization predicts the correct value. Even if the method were not so charmed, there is no point in doing a proper, calculus minimization. The calculus method is *too accurate* given the inaccuracies in the rest of the derivation. Engineers understand this principle. If a bicycle most often breaks at welds in the frame, there is little point replacing the metal between the welds with expensive, high-strength aerospace materials. The new materials might last 100 years instead of 50 years, but such a replacement would be overengineering. To improve a bicycle, put effort into improving or doing without the welds. In estimating the Bohr radius, the kinetic-energy estimate (5.12) uses a crude form of the uncertainty principle, $\Delta p \Delta x \sim \hbar$, whereas the true statement is that $\Delta p \Delta x \geq \hbar/2$. The estimate also uses the approximation $\mathrm{KE} \sim (\Delta p)^2/m$. This approximation contains m instead of $2m$ in the denominator. It also assumes that Δp can be converted into an energy as though it were a true momentum rather than merely a crude estimate for the root-mean-square momentum. The potential- and kinetic-energy estimates use a crude definition of position uncertainty Δx: that $\Delta x \sim r$. After making so many approximations, it is pointless to minimize the result using the elephant gun of differential calculus. The order-of-magnitude method is as accurate as the approximations in the energy.

This method of equating competing terms is **balancing**. We balanced the kinetic energy against the potential energy by assuming that they are roughly the same size. The consequence is that $a_0 \sim \hbar^2/m_\mathrm{e}e^2$. Nature could have been unkind: The potential and kinetic energies could have differed by a factor of 10 or 100. But she is kind: The two energies are roughly equal, except for a constant that is nearly 1. ‘Nearly 1’ is also called **of order unity**. This rough equality occurs in many examples, and you often get a reasonable answer by pretending that two energies (or two quantities with the same units) are equal. When the quantities are potential and kinetic energy, as they often are, you get extra safety: The virial theorem (see Section 5.5) protects you against large errors.

5.3 Numerical evaluation

Having derived

$$a_0 \sim \frac{\hbar^2}{m_\mathrm{e}e^2} \tag{5.27}$$

in two ways, it is time to evaluate it, to put in numbers. For this numerical evaluation, you might be tempted to convert (5.27) to SI units by changing e^2 to $e^2/4\pi\epsilon_0$. You might also think that you need to know or look up $\hbar$ and m_e. You can use this brute-force method, but you can avoid it by using three useful combinations of fundamental constants. The first two are round values that use eV as the unit of energy:

$$\hbar c \simeq 2000\,\text{eV Å}, \tag{5.28}$$

$$m_e c^2 \simeq 500\,\text{keV}. \tag{5.29}$$

$$\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137} \sim 0.01. \tag{5.30}$$

The last combination is the **fine-structure constant**, a dimensionless measure of the strength of the electrostatic force. Dimensionless numbers, such as α, are worth memorizing, because their value is the same in any unit system.

The Bohr radius (5.27) contains e^2 downstairs. If an $\hbar c$ appears upstairs, then the fine-structure constant (5.30) can absorb e^2 and $\hbar c$. The upstairs already contains $\hbar$ (two of them), but c is missing. So multiply by unity in the form c/c:

$$a_0 \sim \frac{\hbar^2}{m_e e^2} \times \frac{c}{c} = \frac{\hbar}{m_e c}\,\frac{\hbar c}{e^2}. \tag{5.31}$$

The quantity in red is α^{-1}. The downstairs still contains m_e, in the form $m_e c$. If only it were $m_e c^2$! The upstairs contains $\hbar$. If only it were $\hbar c$! You can reach both goals by multiplying by unity, again in the form c/c:

$$a_0 \sim \frac{c}{c} \times \frac{\hbar}{m_e c}\,\alpha^{-1} \sim \frac{\hbar c}{m_e c^2}\,\alpha^{-1}. \tag{5.32}$$

You can go from (5.27) to (5.32) by multiplying by c^2/c^2. The individual multiplications by c/c distinguish two related tricks. The first trick is to manufacture α. The second trick is to manufacture $m_e c^2$. When you add enough powers of c to convert every $\hbar$ into $\hbar c$, you usually also convert every m_e into $m_e c^2$. In such cases, the two tricks combine into one, which here is multiplication by c^2/c^2. In the form (5.32), the Bohr radius is easy to evaluate using (5.28) for $\hbar c$, (5.29) for $m_e c^2$, and (5.30) for α:

$$a_0 \sim \frac{2000\,\text{eVÅ}}{5 \cdot 10^5\,\text{keV}} \times 100 \sim 0.5\,\text{Å}. \tag{5.33}$$

The size of hydrogen sets the scale for the sizes of more complex atoms. Hydrogen is the simplest atom; it has one electron, and therefore one energy shell. The second row of the periodic table contains elements with two shells; the third row contains elements with three shells. The most abundant elements on earth (oxygen, carbon, silicon) come from the second and third rows. As a rule of thumb, the diameter of an atom with n shells is n Å, for $n \leq 3$; for $n > 3$, the diameter is still 3 Å, because the extra nuclear charge in those atoms drags the electrons closer, and makes up for the increased number of shells.

The following argument expands on this statement about extra nuclear charge. Consider an atom with atomic number Z; it has Z protons and Z electrons. The outermost electron moves in a potential created by the Z protons and the $Z-1$ other electrons. We cannot easily calculate the charge distribution, so we need to simplify. Imagine that the other electrons orbit inside the outermost electron. This assumption is a crude approximation; it does not account for important facts from quantum mechanics, such as the Pauli exclusion principle,[1] nor does it accurately represent atoms in which two or more electrons are in the outermost shell. However, it is a simple assumption, and has truth in it. What charge distribution does the outermost electron see? It sees a nucleus with charge e (effectively, a single proton): The Z protons and the $Z-1$ electrons almost cancel. An outermost electron orbits a single proton – this configuration is the description of hydrogen. So the environment of the outermost charge is independent of Z, and every large-Z atom is the size of hydrogen. Part of this conclusion is reasonably accurate: that every large-Z atom is a similar size. Part of the conclusion is not correct: that its size is the size of hydrogen. It is incorrect because of the extreme approximation in assuming that every other electron orbits inside the outermost electron, and because it neglects the Pauli exclusion principle. We retain the reasonably correct part, and use $a \sim 3\,\text{Å}$ for a typical atomic diameter or for a small molecular diameter.

5.4 Densities

Atomic sizes determine densities (step 2 in Figure 5.1). An atom is a positive nucleus surrounded by a cloud of negative charge. A solid or liquid contains atoms jammed together. The electron cloud limits how closely the atoms can approach each other. At short distances, the repulsive force between the electron clouds is large because the clouds overlap. At large distances (relative to the Bohr radius), two atoms hardly interact. Between these extremes lies a minimum-energy distance, a, which is the diameter of the electron cloud. So $a \sim 3\,\text{Å}$ is a typical interatomic spacing in a solid or liquid.

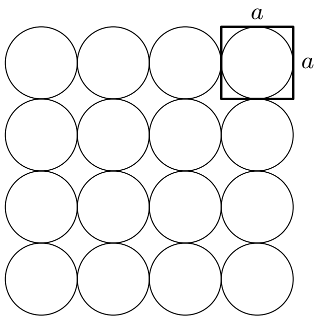

Figure 5.5. *Packing of atoms in a solid or liquid. For clarity, the diagram shows only a two-dimensional slice through the substance. Each atom occupies a cube of side length* $a \sim 3\,\text{Å}$*, or a volume of* $\sim 30\,\text{Å}^3$ *or* $3\cdot 10^{-23}\,\text{cm}^3$*.*

Let A be the atomic mass of the atom; A is roughly the number of protons and neutrons in the nucleus. Although it is called a mass, A is dimensionless. Each atom occupies a cube of side length $a \sim 3\,\text{Å}$ (Figure 5.5), and has mass Am_p. The density of the substance is

$$\rho = \frac{\text{mass}}{\text{volume}} \sim \frac{Am_\text{p}}{(3\,\text{Å})^3}. \tag{5.34}$$

You do not need to remember or look up m_p if you multiply by unity in the form of N_A/N_A, where N_A is Avogadro's number:

$$\rho \sim \frac{Am_\text{p}N_\text{A}}{(3\,\text{Å})^3 \times N_\text{A}}. \tag{5.35}$$

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[1] *Gasiorowicz's text [16] is particularly compact and clear on quantum mechanics and the Pauli principle.*

The numerator is A g, because that is how N_{A} is defined. The denominator is $3\cdot 10^{-23}\,\mathrm{cm}^3 \times 6\cdot 10^{23} = 18$. Instead of remembering m_{p}, you need to remember N_{A}, but N_{A} is more familiar (it arises in chemistry and physics, for example) and it emphasizes the connection between microscopic and macroscopic results, so use it when possible. Thus

$$\rho \sim \frac{A}{18}\,\mathrm{g\,cm^{-3}}. \tag{5.36}$$

Table 5.2 compares this estimate against reality. Most everyday elements have atomic masses between 15 and 150, so the density estimate (5.36) explains why most densities lie between 1 and $10\,\mathrm{g\,cm^{-3}}$. It also shows why, for materials physics, cgs units are more convenient than SI units. A typical cgs density is $3\,\mathrm{g\,cm^{-3}}$, and 3 is a modest number and easy to work with. However, a typical SI density is $3000\,\mathrm{kg\,m^{-3}}$. Numbers such as 3000 are unwieldy. Each time you use it, you have to think, 'How many powers of ten were there again?' We even threw a joker into the pack – water is not an element! – but the density estimate (5.36) is amazingly accurate. Example 5.1 answers a question that may be painfully familiar if you have moved house or apartment.

	$\rho\,(\mathrm{g\,cm^{-3}})$	
Element	*Estimated*	*Actual*
Li	0.39	0.54
H_2O	1.0	1.0
Si	1.56	2.4
Fe	3.11	7.9
Hg	11.2	13.5
Au	10.9	19.3
U	13.3	18.7

Table 5.2. *Actual and estimated densities of solids and liquids. The estimates are from (5.36).*

Example 5.1 *How heavy is a small box filled with books?*

Books are mostly paper; as we reasoned in Section 1.1, paper has the same density as water, so $\rho_{\mathrm{book}} \sim 1\,\mathrm{g\,cm^{-3}}$. In the United States the canonical book box is the small United Parcel Service box. Its volume is $45\,\mathrm{cm} \times 30\,\mathrm{cm} \times 30\,\mathrm{cm} \sim 4\cdot 10^4\,\mathrm{cm}^3$, so its mass is $m \sim 40\,\mathrm{kg}$ – approximately the mass of a light person. These boxes are heavy, and could not be made larger without risking people hurting their backs when they pick up the boxes.

5.5 Binding energy of hydrogen

Part of the order-of-magnitude picture of materials is the spacing between atoms (bond size). Another part is the force or interaction energy between atoms (bond energy): step 3 in Figure 5.1. Following the pattern of Section 5.3, first estimate the **binding energy** of hydrogen, and then extend the understanding to more common and more complex materials.

In hydrogen, the binding energy is the energy required to drag the electron infinitely far from the proton. In more complex atoms, it is the energy to remove all the electrons. We can estimate the binding energy by first estimating the potential energy in hydrogen. The potential energy of an electron and a proton separated by the Bohr radius is

$$\mathrm{PE} \sim -\frac{e^2}{a_0} \sim -\frac{m_{\mathrm{e}}e^4}{\hbar^2}. \tag{5.37}$$

The binding energy is $-E_{\mathrm{total}}$, where E_{total} is the total energy – it includes kinetic as well as potential energy. What is the kinetic energy? The **virial theorem** says that, with a $1/r^n$ force law

$$\langle \mathrm{PE} \rangle = -n\langle \mathrm{KE} \rangle. \tag{5.38}$$

For hydrogen the force is electrostatic, so $n = 2$ and

$$E_{\text{total}} = \text{PE} + \text{KE} = \frac{\text{PE}}{2}. \tag{5.39}$$

The binding energy is

$$E_0 = -E_{\text{total}} = \frac{1}{2}\frac{m_e e^4}{\hbar^2}. \tag{5.40}$$

To evaluate this energy, notice the e^4 upstairs. It suggests manufacturing α^2. So multiply by unity in the form c^2/c^2, to make $(\hbar c)^2$ in the denominator:

$$\begin{aligned} E_0 &\sim \frac{1}{2} m_e c^2 \frac{e^4}{(\hbar c)^2} = \frac{1}{2} m_e c^2 \alpha^2 \\ &\simeq \frac{1}{2} \times 5.1 \cdot 10^5\,\text{eV} \times \left(\frac{1}{137}\right)^2 = 13.6\,\text{eV}. \end{aligned} \tag{5.41}$$

For future reference, we quote useful energy conversions:

$$1\,\text{eV} \simeq 1.6 \cdot 10^{-19}\,\text{J}, \tag{5.42}$$

$$1\,\text{cal} \simeq 4\,\text{J}, \tag{5.43}$$

and

$$\begin{aligned} 1\,\text{eV/molecule} &\simeq 25\,\text{kcal mol}^{-1} \\ &\simeq 100\,\text{kJ mol}^{-1}. \end{aligned} \tag{5.44}$$

The estimate (5.41) is the correct ground-state energy of hydrogen (neglecting relativistic effects such as spin). The many errors in the approximations have, perhaps by luck, canceled. To interpret the result, regroup the factors as

$$E_0 \sim \frac{1}{2} m_e (\alpha c)^2. \tag{5.45}$$

In this form, it is the kinetic energy of an electron with velocity αc. So the fine-structure constant is the velocity of an electron in **atomic units** where charge is measured in units of e, length in units of a_0, and velocity in units of c.

5.6 Molecular binding energies

Covalent and ionic bonds are formed by attractions between electrons and protons; the hydrogen atom is a crude model of this interaction. The main defect of this model is that the electron–proton distance in a hydrogen atom is much smaller than it is in most materials. In most materials, the distance is roughly $a \sim 3\,\text{Å}$, six times larger than the Bohr radius $a_0 \sim 0.5\,\text{Å}$. For covalent and ionic bonds – their binding energy is from the electrostatic attraction between monopoles

– the binding energy is smaller than E_0 by a factor of 6, so $E_{\text{bond}} \sim 2\,\text{eV}$. The factor of 6 occurs because $a \sim 6a_0$ and electrostatic energy scales as $E \propto 1/r$. Scaling E is more direct than is evaluating e^2/a from scratch. It does not clutter the derivation or your thinking with irrelevant information, such as the value of $\hbar$ or α. Table 5.3 lists bond energies. For bonds that include carbon, oxygen, or hydrogen – which have $a \sim 1.5\,\text{Å}$ – we expect $E_{\text{bond}} \sim 4\,\text{eV}$ instead of $2\,\text{eV}$. This expectation is confirmed by the tabulated values.

The **cohesive energy** ϵ_c is the energy required to break the weakest type of bond in an object. In water it is the energy required to remove a water molecule from the liquid into the gas, i.e. to boil it away or evaporate it. In hydrogen it is the energy required to break the electron–proton bond and is given by (5.41). The typical magnitude of ϵ_c indicates why the electron–Volt is a convenient unit in materials physics: Measured in eV, cohesive energies are of order unity.

Bond	*Energy* (eV)
C–C	6.3
H–H	4.5
Na–Cl	4.3
H–O	4.4
Fe–O	4.0
C–H	3.5
Si–Si	3.3
Hg–S	2.2
Cd–S	2.1
Zn–S	2.1

Table 5.3. *Approximate covalent and ionic bond energies. Source:* CRC Handbook of Chemistry and Physics *[37, pp. 9-123 – 9-128]. A typical bond energy is* $4\,\text{eV}$*, which is* $100\,\text{kcal mol}^{-1}$ *in old-style chemical units or* $400\,\text{kJ mol}^{-1}$ *in newer chemical units.*

5.7 Surface tension

From atomic (or molecular) sizes and binding energies, you can now estimate many properties of materials. As an example, we study surface tension (step 4 in Figure 5.1). Water droplets are spherical, because spheres have the smallest surface area for a given volume. This mathematical explanation hides a physical question: Why do water droplets minimize their surface area? The reason is that water has **surface tension**. Surface tension is the energy required to create one unit of new surface, so surface tension has units of energy per area. You can estimate its value using an approximate model. In an order-of-magnitude picture of a liquid, every atom in the interior is bonded to six neighbors: one each in coordinate direction. However, each atom on the surface has only five neighbors. A surface of area A has $N = A/a^2$ atoms. To make such a surface requires breaking N bonds, one for each atom. The cohesive energy is the energy to break all six bonds around the atom, so $E_{\text{bond}} \sim \epsilon_c/3$. To avoid counting each bond twice, we divided by 3 instead of by 6. So the surface energy is

$$E \sim \frac{A}{a^2} E_{\text{bond}} \sim \frac{A\epsilon_c}{3a^2}. \tag{5.46}$$

The surface tension is the energy per area:

$$\gamma \equiv \frac{E}{A} \sim \frac{\epsilon_c}{3a^2}. \tag{5.47}$$

In the exercises, you are asked to estimate the cohesive energy of water from its heat of vaporization. You should find $\epsilon_c \sim 0.5\,\text{eV}$. Using the standard spacing $a \sim 3\,\text{Å}$, we get

$$\begin{aligned}\gamma_{\text{water}} &\sim \frac{0.5\,\text{eV} \times 1.6\cdot 10^{-19}\,\text{J eV}^{-1}}{3 \times (3\cdot 10^{-10}\,\text{m})^2} \\ &\sim 0.3\,\text{J m}^{-2}.\end{aligned} \tag{5.48}$$

Although the basic model is sound, this estimate is too large by a factor of 4, partly because of the crudity of the six-neighbor picture of solids.

Knowing surface tension, you can estimate the size of raindrops (step 5 in Figure 5.1). Falling raindrops feel a drag force that tries to split the drop into smaller droplets. Many small drops have more surface area than one large drop of the same volume. So splitting the drop requires energy. So surface tension resists the splitting force (from drag). At a happy, intermediate size, these effects fight to a draw. You can estimate this size, therefore, by balancing the surface tension (as a force, F_γ) and the drag F_{drag} (Figure 5.6). First estimate the surface-tension force. Surface tension is energy per area, which is also force per length. To obtain a force, think about dimensions. You need to multiply γ by the only length involved: the circumference, which is r except for constant factors. So

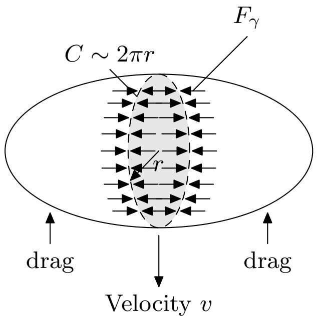

Figure 5.6. Falling raindrop of density ρ_{w} and radius r. The drop has reached terminal velocity v. Surface tension (the facing arrows) F_γ binds the two halves of the drop along the circumference C. Surface tension fights drag, which tries to break the drop. The shaded area shows a cross section of the drop.

$$F_\gamma \sim r\gamma. \tag{5.49}$$

The drag is probably the turbulent drag of (4.54). However, you do not need to calculate the drag, because you already know its magnitude: The drop moves at constant velocity (terminal velocity), so the drag force is its weight. Therefore $F_{\text{drag}} \sim \rho_{\text{w}} r^3 g$, where $\rho_{\text{w}} r^3$ is the mass of the drop. Equating the drag and surface-tension forces gives $\rho_{\text{w}} r^3 g \sim r\gamma$, or

$$r_{\text{max}} \sim \sqrt{\frac{\gamma}{\rho_{\text{w}} g}}. \tag{5.50}$$

For water $\gamma \sim 0.07\,\text{N}\,\text{m}^{-1}$ so the maximum size of raindrops is

$$r_{\text{max}} \sim \sqrt{\frac{0.07\,\text{N}\,\text{m}^{-1}}{10^3\,\text{kg}\,\text{m}^{-3} \times 10\,\text{m}\,\text{s}^{-2}}} \sim 3\,\text{mm}. \tag{5.51}$$

Their terminal velocity is given by the result for turbulent drag:

$$v \sim \sqrt{g r_{\text{max}} \frac{\rho_{\text{water}}}{\rho_{\text{air}}} \frac{2}{c_{\text{d}}}}. \tag{5.52}$$

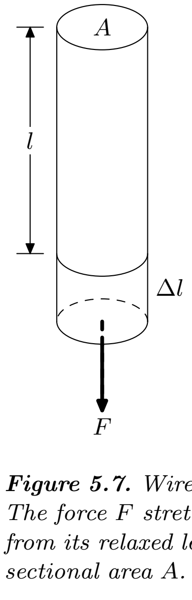

Figure 5.7. Wire stretched by a force. The force F stretches the wire by Δl from its relaxed length l; it has cross-sectional area A.

Here we used (4.56) but including the factor of $c_{\text{d}}/2$ in the drag force. With $\rho_{\text{water}}/\rho_{\text{air}} \sim 1000$ and $c_{\text{d}} \sim 0.5$ for spherical droplets, the terminal velocity is $v \sim 10\,\text{m/s}$, which is the speed of slow driving or fast sprinting. This terminal velocity seems right: If you drive slowly in a rainstorm, the drops strike at roughly a 45° angle.

5.8 Stiffness

Next in our journey through materials physics (step 6 of Figure 5.1) is **stiffness**: the resistance of a material to stretching, squeezing, or bending. The quantity $\mathcal{M}$ is the **elastic modulus**. It is analogous to the spring constant but more useful. To see why, imagine a wire (Figure 5.7) stretched by a force. Its spring constant determines the extension: $F = k_{\text{w}}\Delta l$. A thicker wire (of the same material) – with, say, area $4A$ – has a correspondingly larger spring constant: $k_1 = 4k_{\text{w}}$. A longer wire – with, say, length $2l$ (and the same area) – has a correspondingly smaller spring constant: $k_{\text{w}}/2$. The spring constant is not a property of the wire alone: It depends on the wire's length and area. The elastic modulus fixes this problem: It depends only on the substance. Elastic moduli are intensive quantities, like resistivity or conductivity (see Section 4.6) and share their virtues.

How can you define such a quantity macroscopically, in terms of lengths, areas, and forces? One dependence is on area: k/A is independent of area. The other is on length: $k_{\text{w}}l$ is independent of length. The combined quantity $k_{\text{w}}l/A$ is independent of area and length. In terms of this quantity, $F = k_{\text{w}}\Delta l$ becomes

$$F = \left(\frac{k_{\text{w}}l}{A}\right)\frac{A}{l}\Delta l, \tag{5.53}$$

or

$$\underbrace{\frac{F}{A}}_{\sigma} = \underbrace{\frac{k_{\text{w}}l}{A}}_{\mathcal{M}}\underbrace{\frac{\Delta l}{l}}_{\epsilon}, \tag{5.54}$$

where σ is stress (or pressure), $\mathcal{M}$ is elastic modulus, and ϵ is fractional change in length, or **strain**, which is the dimensionless measure of extension. So

$$\mathcal{M} = \frac{k_{\text{w}}l}{A}. \tag{5.55}$$

We can count bond springs to find a microscopic expression for $\mathcal{M}$. The wire of Figure 5.7 is a bundle of filaments, where each filament is a line of atoms. Each filament has $N_{\text{l}} \sim l/a$ springs; the spring constant of a filament is therefore reduced by l/a compared to the spring constant of a bond. The wire contains $N_{\text{f}} \sim A/a^2$ filaments; this factor increases the spring constant of the wire compared to the spring constant of a filament. The spring constant of the wire is then

$$k_{\text{w}} = k\frac{N_{\text{f}}}{N_{\text{l}}} \sim k\frac{A}{la}, \tag{5.56}$$

where k is the spring constant of one bond. With this expression, the macroscopic definition of elastic modulus (5.55) becomes

$$\mathcal{M} \sim \frac{k}{a}. \tag{5.57}$$

To estimate the spring constant k, think about its dimensions. You can make an energy with kl^2, where l is a length. Here a natural length is the bond length, and a natural energy is the cohesive energy ϵ_{c}. So perhaps $k \sim \epsilon_{\text{c}}/a^2$. This estimate is very rough, but is of the right order of magnitude. With it, the modulus (5.57) becomes

$$M \sim \frac{\epsilon_{\text{c}}}{a^3}. \qquad (5.58).$$

Now you can estimate a typical modulus:

$$\begin{aligned}\mathcal{M} &\sim \frac{\epsilon_c}{a^3} \sim \frac{2\,\mathrm{eV}}{(3\,\text{Å})^3} \sim \frac{2\times 1.6\cdot 10^{-19}\,\mathrm{J}}{3\cdot 10^{-29}\,\mathrm{m}^3} \\ &\sim 10^{10}\,\mathrm{J\,m^{-3}}.\end{aligned} \tag{5.59}$$

This estimate for $\mathcal{M}$ is reasonable for many materials (Table 5.4).

Substance	$\mathcal{M}$ $\left(10^{10}\,\mathrm{J\,m^{-3}}\right)$
Graphite, ∥ to planes	100
Diamond	44
Steel	20
Ash (white)	12
Cu	11
Al	7.9
Glass	6
Graphite, ⊥ to planes	3.5
Pb	1.8
Ice (−5 °C)	0.9

Table 5.4. *Elastic moduli. There are numerous elastic moduli. We list for each substance the Young's modulus, which is a combination of the shear modulus K and the bulk modulus B. Note how strongly $\mathcal{M}$ for graphite depends on direction, the result of the layered structure of graphite. If $\mathcal{M}_\perp$ were as high as $\mathcal{M}_\parallel$, then lead pencils would be diamond styluses, and diamonds would be a lot cheaper. Source:* Smithsonian Physical Tables *[15].*

In a gas at pressure P, the speed of sound is roughly

$$c_s \sim \sqrt{p/\rho}, \tag{5.60}$$

where p is the pressure and ρ is the density. The sound-speed formula (5.60) is the only dimensionally correct combination of c_s, p, and ρ. For a solid or liquid, hope for a similar relation, after replacing pressure with an analogous quantity. The dimensions of pressure are force per area, which is also energy per volume. In materials, a natural volume is the interatomic volume a^3. A natural energy is the cohesive energy ϵ_c. Their combination ϵ_c/a^3 is a reasonable analog of pressure. Oh, wait, that's the same as the modulus in (5.58). The speed of sound is then

$$c_s \sim \sqrt{\frac{\mathcal{M}}{\rho}} \sim \sqrt{\frac{\epsilon_c}{\rho a^3}} \sim \sqrt{\frac{\epsilon_c}{m}} \tag{5.61}$$

where $\rho a^3 \sim m$ is the mass of an atom.

To estimate a typical sound speed, use (5.59). A 'typical' solid has atomic mass, say, 50; from (5.36), it has density $\rho \sim 3\,\mathrm{g\,cm^{-3}}$, so

$$c_s \sim \sqrt{\frac{\mathcal{M}}{\rho}} \sim \sqrt{\frac{10^{10}\,\mathrm{J\,m^{-3}}}{3\cdot 10^3\,\mathrm{kg\,m^{-3}}}} \sim 1.7\,\mathrm{km\,s^{-1}}. \tag{5.62}$$

This estimate is reasonably accurate (Table 5.5).

Substance	$v\,(\mathrm{km\,s^{-1}})$
Glass	5.5
Steel	5.0
Brick	3.7
Pine	3.3
Water	1.5
Hg	1.5
Cu	3.6
Cork	0.5
Granite	4.0

Table 5.5. *Speed of sound at room temperature. The generic estimate is $v \sim 1.7\,\mathrm{km\,s^{-1}}$. It is a useful rule of thumb; but the rule of thumb underestimates the speed in metals and stiff nonmetals such as glass, and overestimates the speed in soft substances such as cork. Source:* Smithsonian Physical Tables *[15, p. 306].*

5.9 Strength

How strong are materials? To break a perfect material (for example, diamond or carbon filament with no flaws), we would have to apply a stress $\sim \mathcal{M}$. Most materials break long before the stress reaches $\mathcal{M}$, because flaws in their crystal structure concentrate stress, so locally the stress may reach $\mathcal{M}$ even if the global stress is much less than $\mathcal{M}$. A typical breaking stress (or yield stress) is between $0.001\mathcal{M}$ and $0.01\mathcal{M}$. The dimensionless factor is the yield or breaking strain, ϵ_y, which we list in Table 5.6. A typical yield stress is $10^8\,\mathrm{J\,m^{-3}}$. We now apply these estimates to mountain climbing.

How much energy does it take to climb the tallest mountain on a planet? First determine the height of such a mountain (Figure 5.8). The strength of the rock limits its height. The mass of the top conical block is ρh^3, where ρ is the density of rock; it produces a force $F \sim \rho g h^3$. The resulting stress is

$$\sigma \sim \frac{F}{A} \sim \frac{\rho g h^3}{h^2} = \rho g h. \tag{5.63}$$

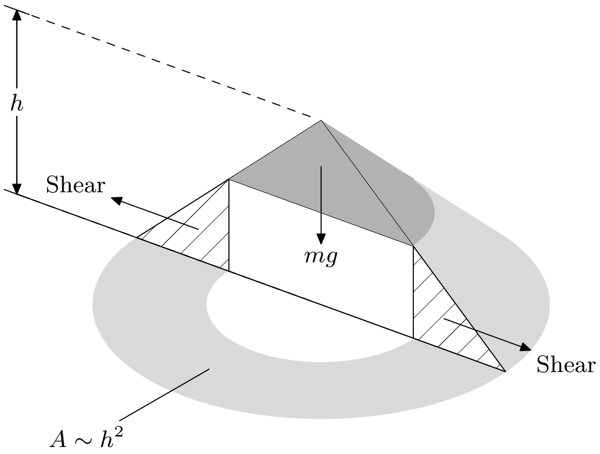

Figure 5.8. *Mountain. This approximate mountain is conical, with 45° slopes. The shaded top block, which has mass m, squeezes the hatched chunk (whose footprint is the lightly shaded ring). When the mountain gets too high, the squeeze becomes greater than the mountain rock can withstand.*

The strain is

$$\epsilon = \frac{\sigma}{\mathcal{M}} \sim \frac{\rho g h}{\mathcal{M}}. \tag{5.64}$$

For rock, $\mathcal{M} \sim 10^{11}\,\mathrm{J\,m^{-3}}$. When $\epsilon \sim \epsilon_{\mathrm{y}}$, the rock yields, and the mountain shrinks until ϵ falls below ϵ_{y}. Therefore, the maximum height for a mountain on the earth is

$$h_{\mathrm{max}} \sim \frac{\mathcal{M}\epsilon_{\mathrm{y}}}{\rho g}. \tag{5.65}$$

For a typical rock, $\epsilon_{\mathrm{y}} \sim 10^{-3}$ and $\rho \sim 3\,\mathrm{g\,cm^{-3}}$, so

$$h_{\mathrm{max}} \sim \frac{10^{11}\,\mathrm{J\,m^{-3}} \times 10^{-3}}{3\cdot 10^{3}\,\mathrm{kg\,m^{-3}} \times 10\,\mathrm{m\,s^{-2}}} \sim 3\,\mathrm{km}. \tag{5.66}$$

Substance	ϵ_{y}
Steel	0.005
Cu	0.002
Al	0.001
Rock	0.001

Table 5.6. *Approximate breaking strain ϵ_{y}.*

This estimate neglects many numerical factors. Not surprisingly, many mountains – such as Everest, with $h \sim 10\,\mathrm{km}$ – are larger than our estimated maximum. (Perhaps Everest is made of extra-hard rock, such as granite.) However, the result is surprisingly accurate considering the approximations made.

Is it more difficult to climb the highest mountains on Mars, or Mercury, or the asteroid Ceres, compared to the highest mountains on the earth? We assume that all astronomical objects are made of the same type of rock. The energy required to climb to the top is

$$E \sim mgh_{\mathrm{max}} \sim \frac{m\mathcal{M}\epsilon_{\mathrm{y}}}{\rho}, \tag{5.67}$$

where m is the mass of the climber. This energy is independent of g: Strong gravity makes it hard to climb mountains, but it also reduces their height, and the two effects cancel. For a typical climber, $m \sim 50\,\mathrm{kg}$, so

$$\begin{aligned} E &\sim \frac{50\,\mathrm{kg} \times 10^{11}\,\mathrm{J\,m^{-3}} \times 10^{-3}}{3\cdot 10^{3}\,\mathrm{kg\,m^{-3}}} \\ &\sim 1.7\cdot 10^{6}\,\mathrm{J} \\ &\sim 400\,\mathrm{kcal}, \end{aligned} \tag{5.68}$$

or one chocolate bar (at perfect efficiency). People do not climb with perfect efficiency; no one could climb Everest on 1 or even 10 chocolate bars.

How high are these mountains? The energy to the climb the mountains does not depend on g, but the maximum height does. As we found in Section 2.2, $g \propto R$. So $h_{\max} \propto R^{-1}$, and the fractional size of mountains $h_{\max}/R$ is $\propto R^{-2}$. For the earth, $h_{\max} \sim 3\,\mathrm{km}$ and $R_{\oplus} \sim 6000\,\mathrm{km}$, so $h/R_{\oplus} \sim 10^{-3}$. For an asteroid of radius $R \sim R_{\oplus}/30$ (roughly 200 km), the fractional height would be roughly 1. This estimate is reasonably accurate. Asteroids with $R \sim 200\,\mathrm{km}$ have mountains roughly their own size; these mountains make the asteroids look significantly oblate. In Section 22.1, you estimate mountain heights on white dwarfs, which have a strong gravitational field, which crushes mountains, and stiff matter, which resists gravity. The mountain example concludes our journey through the mechanical properties of materials.

5.10 What you have learned

- *Atomic theory:* All matter is made of atoms. They attract at moderate distances and repel when squeezed together.
- *Atomic sizes:* Atoms and simple molecules (like water) have a diameter of a few Å.
- *Densities:* Most liquids and solids have densities between 1 and $10\,\mathrm{g\,cm^{-3}}$.
- *Binding energies:* Typical covalent and ionic bond energies are a few eV.
- *Quantum mechanics:* Many results of quantum mechanics arise simply by adding $\hbar$ to the list of relevant variables for dimensional analysis. Quantum mechanics, through $\hbar$, introduces a new momentum scale, the uncertainty momentum Δp. This momentum produces a confinement energy $(\Delta p)^2/m$.
- *Balancing:* Many physical systems contain two competing processes. For example, one energy (in hydrogen, electrostatics) competes with another energy (in hydrogen, the uncertainty energy). The energies are often equal near the minimum-energy state. Look for these competitions.
- *Don't overengineer:* There is little point computing one part of a result to six decimal places of accuracy when equally important parts are accurate only to 10 percent.

5.11 Exercises

▶ **5.25** *Water*
Estimate the cohesive energy of water, which (explain why!) is also its heat of vaporization.

► **5.26** *Water again*
Use your estimate to estimate the bulk modulus of water and the speed of sound.

► **5.27** *Steel*
What is the maximum length of a steel wire before it breaks under its own weight?