

Estimating gas mileage: An example of order-of-magnitude physics

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Abstract. I discuss how to estimate the gas mileage of a car. This discussion, which covers air resistance and Reynolds numbers, describes one way to introduce dimensional analysis and order-of-magnitude physics into introductory physics (if only the syllabus would allow it). It is part teacher’s guide and part textbook chapter – I hope not the worst parts of each.

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1 The problem

Can we predict the gas mileage for a car (in miles per gallon)? We can begin the discussion by asking students why a car requires gasoline. Where does the energy go? Eventually students say: some sort of resistance. What kind? Air resistance. Here is a chance to teach a principle of science: *Test your ideas*. Have confidence in your ideas, but not too much; the arms-control negotiator says ‘trust, but verify’. We test our model – that air resistance consumes most of the power – by calculating whether air resistance accounts for the gasoline consumed.

How large is air resistance? Before students can answer ‘how large’, they must think about how to measure air resistance. Is it a force, a pressure, an energy? Gasoline provides energy, so let’s compute the energy consumed by air resistance, and equate it to the energy provided by one gallon of gasoline. Energy is force times distance: $E_{\text{drag}} = Fd$, where F is the air-resistance force and d is distance traveled. If E_{gallon} is the energy provided by one gallon of gasoline, and $E_{\text{gallon}} \sim E_{\text{drag}}$, then $d = E_{\text{gallon}}/F$ is the distance a car can travel on that gallon. The problem breaks into two computations: the air-resistance force and the energy available from 1 gallon of gasoline. This breakdown is an example of *divide-and-conquer* reasoning, a frequent technique in order-of-magnitude physics and in everyday thinking.

2 Air resistance

How can we compute the air-resistance force? We can scare students by writing down the Navier–Stokes equations from fluid mechanics, as a vector equation with gradients and dot products:

$$(\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}. \quad (1)$$

If the plethora of symbols confuses students, consider it a job well done. Now we can increase the tension, when we tell them that these equations are vector shorthand for three coupled nonlinear partial-differential equations:

$$\begin{aligned} \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) + \frac{\partial v_x}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right), \\ \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) + \frac{\partial v_y}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right), \\ \left(v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) + \frac{\partial v_z}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right). \end{aligned} \quad (2)$$

To find the force, we solve these equations for the pressure, p . We’ll solve this problem after studying partial-differential equations for three years. Students with any imagination by now tremble a bit, and are receptive to a simpler method. When they hear that we have not listed the complete set of equations – the set (2) leaves out the continuity equation – students are distressed. Estimation plus dimensional analysis is a simple and quick method for finding the drag force.

2.1 Choosing relevant quantities

These approximate methods, although mathematically simple, require physical imagination. To stimulate the imagination, we begin by deciding which features of the problem determine the air resistance. Air, like any fluid, resists the motion of an object moving through it. This description suggests two categories of relevant features: characteristics of the car and of the air.

The car's speed, v , determines drag. Gales can knock over trees; gentle breezes cannot. This argument about moving air might cause students to wonder: Weren't we talking about still air and a moving car? We were, but the two descriptions – moving air with stationary car, or stationary air with moving car – are equivalent. Therefore, we can reason about a reference frame in which a stationary car is buffeted by a wind (of speed v), and transfer that reasoning to the frame where the car moves through still air.



Figure 1. Two cars, one tailgating the other (view from the side).

The car's size also matters: Large cars feel more drag than small cars do. How should we measure size? Length should not affect air resistance, as the following thought experiment suggests. Imagine two cars, one tailgating the other (Figure 1). The rear car feels almost no drag; some cyclists try a related dangerous activity: riding behind a truck to reduce air resistance (as in the movie *Breaking Away* (1979)). In the limit of zero tailgating distance (Figure 2), the two cars merge into one long car. The long car has the same drag as one short car.

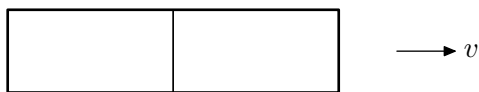


Figure 2. No distance between the cars (view from the side). The two short cars meld into one long car (heavy outline) that feels the same air resistance as one short car feels.

This last statement is surprising, but you can perform a demonstration to convince yourself and your students. Hold a book in one hand and a piece of paper in the other hand, at say chest height. Ask which object will hit the ground first. Most predict that the book hits first. Don't drop anything, for that would only reward rash responses! Instead place the paper under the book (choose the paper so that it's slightly smaller than the book) and then drop the combined object. They hit at the same time. The audience will protest that you cheated, because 'the book is forcing the paper down'. Agree with the criticism: Offer to put the paper on top of the book and drop the book and paper. However, ask for predictions first: What will happen? The two objects fall as one. Many dubious explanations will be offered, including that the book 'sucks the paper downward'. But the simplest explanation is also the correct one: The top object (the paper) feels no air resistance, so it falls like a stone. The bottom object (the book) feels air resistance, but being heavy the drag hardly affects it on the short journey to the floor (it too falls like a stone). Similarly, in Figure 2, the second car experiences no air resistance, so the double car feels the same drag as one short car does. Therefore, car length should not affect air resistance.

Physicists experiment in their minds all the time. Some theorists are, like me, limited to thought experiments, where equipment is cheap and clumsiness no handicap. Experimentalists also use thought experiments; how else could they design a real experiment? Skill in designing and using such experiments is one of the most valuable lessons that physics can teach. It develops the student's imagination. I therefore interleave thought experiments throughout this discussion.

To decide how width affects air resistance, consider a related thought experiment: two cars traveling side by side (Figure 3). Each car feels the same resistance as one car. In the limit that

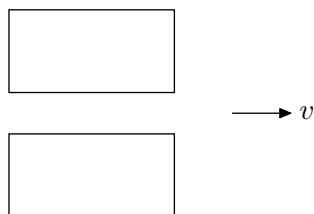


Figure 3. Two cars traveling side by side (view from above).

the two cars are adjacent, the double-width car feels twice the resistance of one single-width car. So resistance should be proportional to width. A thought experiment with one car traveling above the other suggests that resistance should be proportional also to height. So a reasonable measure of size is height times width, or frontal area A . The analysis of the the relevance of area shows students two examples of thought experiments. Three examples are a charm: Students understand an idea after seeing three examples that use it. Read on to see the third example.

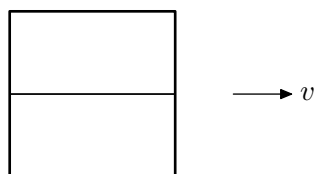


Figure 4. Limit of two cars traveling side by side (view from above). The two cars meld into one wide car (heavy outline) that experiences twice the air resistance that one thin car feels.

The density of the fluid also determines drag. If students don't realize that the density of air matters, ask why it is tiring to run in a swimming pool: because water is thick and air is thin. When we discuss density of the fluid, students naturally wonder whether the density of the car affects air resistance. To answer this question, we can use another thought experiment – the third! Imagine a car with its windows sealed, traveling at 60 mph. Stop the car, invite four large friends into the car – preferably friends raised on steak, potatoes, and growth hormone – and speed up to 60 mph. The density of the car increases, but does the air know about the contents of the car? No. To the air, the car is a black box: Its contents are invisible. The air knows only the car's speed and the shape and texture of its surface. So the density of the car should not affect air resistance.

Our thought experiments tell us that the drag force, F , depends on ρ , v , and A . It could also depend on viscosity, a reasonable proposal since viscosity is the only mechanism of energy loss in the problem, so it is the only source of drag. If the viscosity is exactly zero, then the drag is also zero. However, as long as the viscosity is not zero, the drag depends only slightly on the viscosity. The explanation is tricky, and the simplest route around this obstacle is to say, 'Trust me for now that the viscosity does not matter. We'll derive our result, then do an experiment at the end to check whether it is reasonable, and thereby check whether I deserve to be trusted on this point.' For readers of a less trusting disposition, jump ahead to [Sections 2.4](#) and [2.5](#), which discuss the relative importance of viscous and inertial drags, and justify the neglect of viscosity.

We often tell students that dimensions are part of a physical quantity, rather than an extra, like salt, to add according to taste. But students do not understand why we exhort them on this point. Here we can show them: finding the drag merely by requiring that F have dimensions of force.

2.2 Dimensions of each quantity

What are the dimensions of each variable? Students know that force can be measured in Newtons, but they often do not realize what dimensions a Newton contains. So we remind them that any valid equation for force, such as $F = ma$, determines the dimensions:

$$[F] = \text{MLT}^{-2}. \quad (3)$$

The dimensions of speed and area cause no trouble:

$$\begin{aligned} [v] &= \text{LT}^{-1} \\ [A] &= \text{L}^2. \end{aligned} \quad (4)$$

Nor should the dimensions of density:

$$[\rho] = \text{ML}^{-3}. \quad (5)$$

But some students think that density is volume per mass. They memorized the phrase ‘mass per volume’ badly, and did not learn the important idea: that an ice cube and an iceberg have the same density, that density is *intensive*. When we discuss the dimensions of density, we can distinguish intensive quantities, such as density and temperature, from extensive quantities, such as mass and heat.

2.3 Looking for the right combination

How can we combine these variables into a quantity with the dimensions of force? When many teach what they call dimensional analysis, they show students how to set up and solve linear equations in order to find the right combination: count powers of mass, length, and time in each variable – so each variable becomes a three-dimensional vector in the space of dimensions – and ask what linear combination of ρ , v , and A vectors makes a force vector. This problem is equivalent to solving a system of linear equations. I like reasoning using the space of dimensions; I should use it to argue intuitively for the Buckingham Pi theorem (quoted without proof in [3, Chapter 3] and used many times in the rest of the document; see Buckingham’s paper [2] for the original statement and proof). But solving the linear equations is pointless. It is a brute-force method that teaches the student little except how to solve linear equations. If a problem is so complicated that we must solve linear equations to find the right combination, then we have too many variables; dimensional analysis will not save us. We need first to simplify the list of variables by using additional physical arguments.

Instead of solving linear equations, we can teach a quick and elegant method. Force contains one power of mass; the only other variable that contains mass is ρ , which also contains one power of mass. So F must be proportional to ρ . Now the problem simplifies: How to combine v and A into F/ρ , which has dimensions of L^4T^{-2} . Apply the same trick to time: F/ρ contains time as T^{-2} , and only the speed has time in it. The speed contains time as T^{-1} , so $F/\rho \propto v^2$. The problem is now even simpler: What do we do to A to make a quantity with the dimensions of $F/\rho v^2$? The dimensions of A and $F/\rho v^2$ are the same, so $F/\rho v^2 A$ is dimensionless. This method of constraints is subtle (it substitutes thought for mindless calculation), but reasoning with constraints is valuable for analyzing complicated problems and is worth teaching. In finding the drag formula, students use the method twice with slight variations. Repetition teaches, but repetition with variation teaches more. (Polyà [4] points out that Mozart, in his piano concertos, did not merely repeat the theme; rather, Mozart restated it with variations.)

Since $F/\rho v^2 A$ is dimensionless, it must be a constant. Voilà: Drag force is proportional to $\rho v^2 A$, a result that we have found without solving any differential equations. Earlier we promised that

Could F be $7000\rho v^2 A$ or $\rho v^2 A/1000$? Sure; our method does not tell us the constant. To find the constant, we would have to solve Navier–Stokes equations (2).

This property is general. When you solve a differential equation, you learn only a dimensionless constant; the rest of the solution – the functional form – is determined by physical constraints, the same constraints that determine the form of the differential equation. The simplest method tells you the most important information; Murphy’s law is not often violated, but when it is, we should be grateful! Differential equations are difficult; physical arguments we can teach.

Let’s analyze free fall, the first problem that students solve with differential equations. How long does a rock take to fall from a height of 10 m (roughly three storeys)? The time depends on the strength of gravity, g , and the height, h . How can g and h combine into a quantity with dimensions of time? There is only one way: $t \sim \sqrt{h/g}$. We can find that expression using the method of constraints. The input variables h and g each contain one power of length, and the fall time contains no length, so t must be a function of the ratio h/g :

$$t = f(h/g). \quad (6)$$

To decide on the functional form, look at the powers of time: h/g contains T^2 , so

$$t = \sqrt{h/g},$$

except for a dimensionless constant. The fall time from three storeys is roughly

$$t \sim \left(\frac{10 \text{ m}}{10 \text{ m s}^{-2}} \right)^{1/2} = 1 \text{ sec}.$$

The differential equation for the position of the object is

$$\frac{d^2 x}{dt^2} = g, \quad (7)$$

where x is the distance traveled since release and t is the time since release. The solution, $x(t) = gt^2/2$, tells us that the object falls a distance h when $t = \sqrt{2h/g}$. The order-of-magnitude analysis left out a dimensionless factor of $\sqrt{2}$. In an order-of-magnitude analysis, we *hope* that the missing constant is close to unity, and often it is. It is worth hoping: Solving a differential equation is much harder than fiddling with dimensions and performing thought experiments.

We now test our conclusion that $\rho v^2 A$ a reasonable expression for drag force. Drag should increase as density increases, as speed increases, or as area increases. Our expression has these properties. This test suggests an alternative method that we could have used to determine the drag force – an alternative worth using if students find the constraint method too tricky. Drag force should increase with speed, density, and area. So let’s try the formula: $F \sim \rho v A$. The dimensions of $\rho v A$ are MT^{-1} . The dimensions of force are MLT^{-2} , so our expressions lacks a factor of LT^{-1} . One more power of v fixes this problem, and we find that $F \sim \rho v^2 A$.

2.4 Stokes' law

What if a student looks in her textbook and finds Stokes' law for a sphere:

$$F = 6\pi\rho\nu vr, \quad (8)$$

where ν is kinematic viscosity of the fluid and r is the radius of the sphere. Why didn't our argument discover Stokes' law? This question is excellent. If a student does not raise the question, we can raise it ourselves. A simple answer is that throwing out viscosity makes it impossible to discover Stokes' law. But let's pretend that we didn't throw out viscosity. In discussing Stokes' law, we get an excuse to discuss viscosity and to compare the relative sizes of the inertial ($\rho v^2 A$) and Stokes' drag forces. Their ratio is the simplest comparison:

$$\frac{\text{inertial drag force}}{\text{Stokes' drag force}} \sim \frac{\rho v^2 A}{\rho \nu v r} \sim \frac{vr}{\nu}, \quad (9)$$

where we have estimated the area A as r^2 . This ratio is dimensionless, and is therefore a valuable quantity. It is the *Reynolds number*, commonly denoted Re , and is a measure of the flow speed (or, equivalently, of the object's speed). Speed? We divided forces; where did speed enter? In the expressions for the drag forces. The Reynolds number turns out to be proportional to v . Alone v cannot measure speed, because v is not dimensionless; its value depends on the system of units. I walk at 3 mph. To make the speed seem slow, I can quote it as

$$\begin{aligned} v_{\text{walk}} &= 1.5 \cdot 10^{-3} \text{ km sec}^{-1} \\ &= 1.5 \cdot 10^{-9} \text{ parsecs yr}^{-1}. \end{aligned} \quad (10)$$

To make the speed seem fast, I can quote it as

$$\begin{aligned} v_{\text{walk}} &= 5 \cdot 10^4 \text{ km yr}^{-1} \\ &= 5 \cdot 10^{21} \text{ \AA century}^{-1}. \end{aligned} \quad (11)$$

This example illustrates an important principle: **No quantity with dimensions is big or small intrinsically.** Is 5 kg a large mass? For a bacterium, yes; for an elephant, no. A quantity with dimensions must be compared to another, *relevant* quantity with the same dimensions; dividing the two quantities results in a dimensionless number, whose value is independent of the system of units. In searching for a relevant comparison, students explore a problem and connect what they discover to their other knowledge. If students had this habit, they would pause before writing down whatever number appears on their calculator display. An inclined plane with a height of 10^{-7} m or a charge of 10^7 C would make students suspect a mistake.

A simple explanation of the Reynolds number is the ratio of inertial and Stokes' drag expressions, as shown in (9). This explanation is slightly misleading. At high Reynolds number, the Stokes' drag expression does not apply; at low Reynolds number, the inertial drag expression does not apply. There's no regime where both expressions apply; taking their ratio is physically slightly misleading. But it is a reasonable way to produce a dimensionless number.

As an alternative explanation, the Reynolds number is the ratio of the object's speed and $v_{\text{diffuse}} = \nu/r$, the speed at which momentum diffuses. Kinematic viscosity, ν , is the diffusivity of momentum; momentum therefore diffuses across an object of size r in time $t \sim r^2/\nu$ (as a dimensional argument suggests). From the length r and the time t , we can form a speed:

$$v_{\text{diffuse}} = \frac{r}{t} \sim \frac{r}{r^2/\nu} = \frac{\nu}{r}, \quad (12)$$

which it is natural to call the diffusion speed.

2.5 Reynolds number

Students can estimate the Reynolds number for various flows, and we can discuss the consequences (oily flow for $Re \ll 1$, turbulent flow for $Re \gg 1$), and show the beautiful pictures from *An Album of Fluid Motion* [7] or *A Gallery of Fluid Motion* [6].

For example, walking across a room,

$$v \sim 200 \text{ cm s}^{-1}, \quad \nu \sim 0.2 \text{ cm}^2 \text{ s}^{-1}, \quad \text{and} \quad r \sim 100 \text{ cm}, \quad (13)$$

so

$$Re \sim \frac{200 \text{ cm s}^{-1} \times 100 \text{ cm}}{0.2 \text{ cm}^2 \text{ s}^{-1}} \sim 10^5. \quad (14)$$

Or, running in a swimming pool:

$$v \sim 100 \text{ cm s}^{-1}, \quad \nu \sim 10^{-2} \text{ cm}^2 \text{ s}^{-1}, \quad \text{and} \quad r \sim 100 \text{ cm}, \quad (15)$$

so

$$Re \sim \frac{100 \text{ cm s}^{-1} \times 100 \text{ cm}}{10^{-2} \text{ cm}^2 \text{ s}^{-1}} \sim 10^6. \quad (16)$$

I have quoted quantities in cgs units rather than in the more common SI (mks) units, so that students see the arbitrariness of unit systems and do not become wedded to a single system.

The only tricky part in the preceding estimate is determining r (are you a sphere?). But we need only an approximate Reynolds number, so an approximate measure of our size is accurate enough for this estimate. This Reynolds number is much greater than unity – a convenient dividing line between fast and slow flows – so the flow is fast. Experiments show that for Re greater than roughly 1000, flow is turbulent. Because air is invisible, we do not appreciate the turbulence that we generate merely by walking, but physics increases the power of our imagination. The Reynolds number in this example is so large that we expect most everyday flows to be turbulent as well.

Another example: a paramecium swimming in pond water. Students can estimate the speed by putting a drop of pond water under the microscope and noting how long it takes the little beast to cross the field of view. I shall make a rough estimate here, based on hazy memories of school biology. At 1000-fold magnification, a paramecium looks 1 cm long, the field of view looks 15 cm wide, and the paramecium swims across it in perhaps 15 sec. I had originally written 30 sec, but I am hardly confident of either value, so I might as well use the numerically convenient value of 15 sec. The ingredients of the Reynolds number are

$$r \sim 10^{-3} \text{ cm}, \quad v \sim 10^{-3} \text{ cm s}^{-1}, \quad \text{and} \quad \nu \sim 10^{-2} \text{ cm}^2 \text{ s}^{-1}, \quad (17)$$

so the Reynolds number is

$$Re \sim \frac{10^{-3} \text{ cm s}^{-1} \times 10^{-3} \text{ cm}}{10^{-2} \text{ cm}^2 \text{ s}^{-1}} \sim 10^{-4}. \quad (18)$$

The flow is excruciatingly slow and viscous; to the paramecium, water is a thick, viscous liquid, the way cold honey or corn syrup is to us. Purcell's article on 'Life at low Reynolds number' [5], a beautiful discussion of this point, is one that we and our students can enjoy.

For everyday flows, inertial drag is the important drag, which explains why we won't worry about Stokes' drag for gas mileage (it turns out that the Stokes'-drag formula is valid only for $Re \ll 1$, and the inertial-drag formula only for $Re \gg 1$).

2.6 Checking the expression for inertial drag

Now we can return to the inertial drag force. We have already checked the expression theoretically, when we verified in [Section 2.3](#) that the form was reasonable. We can also check it experimentally, by *putting in numbers*. We get another chance to reinforce the moral: Doubt, question, check, never trust yourself completely. Now that we are about to do arithmetic, I tell students that ‘calculators rot their brain’. I forbid my students from using them; they would be able to calculate to one digit without a calculator, except that calculator use has atrophied their numerical sense. So students need to practice – they need to put in numbers – to recover their feel for numbers.

In what situation can we test the formula for the drag force? Eventually we test it when we estimate the gas mileage, but the gas-mileage example is not the ideal test: We want to use the formula to test also whether air resistance is the main contribution to gas mileage. So we ought to test the formula in another example – to gather independent evidence. Ideally, this new example would use students’ knowledge of their everyday world. Students learn little if we show them how the drag formula constrains, for example, the design of supersonic transports.

Instead we might analyze why running in a swimming pool is so exhausting, and how fast people can run in a swimming pool. The speed is limited by the power that a person can generate; this power goes to fighting drag. How much power can a person generate? It depends on the person, but let’s ask about a typical person. The power is roughly a few hundred watts – as a student may know if at a science museum she has tried to light a bulb using a bicycle. So $P_{\text{avail}} \sim 300 \text{ W}$. Always ask, and get students to ask: **How reasonable is that number?** One way to judge it is to compare it to another, similar power: the horsepower, roughly 750 W . So a person, with, say, one-fifth the mass of a horse and presumably one-fifth the muscle mass too, can put out almost one-half the power? Maybe the 300 W is an overestimate, but on the other hand, humans have lots of muscle in their legs, whereas horses have – for their greater weight – relatively spindly legs. So maybe a hard-cycling human can generate more power per mass than a horse can, and the 300 W is roughly right. Either way, it’s not far off so let’s use the value.

The power consumed by drag is the drag force times the person’s speed or $\rho v^3 A$. The estimated speed is

$$v \sim \left(\frac{P_{\text{avail}}}{\rho A} \right)^{1/3}. \quad (19)$$

Students now get another chance to put in numbers. The density of water is easy: 10^3 kg m^{-3} . My frontal area – divide-and-conquer reasoning once again – is $2 \text{ m} \times 0.5 \text{ m}$ or 1 m^2 . To estimate an area, split the problem in two: into estimating length and estimating height. Arons, in *Teaching Introductory Physics* [1, p. 12], discusses how students ‘know’ the area of a square or of a circle, but not of an irregular figure, for which no formula is available; the notion that area is length times width, even when the length and width are not precisely defined, does not occur to students. An order-of-magnitude area estimate, such as for a person’s frontal area, teaches this idea.

We now put the pieces together to find v :

$$v \sim \left(\frac{300 \text{ W}}{10^3 \text{ kg m}^{-3} \times 1 \text{ m}^2} \right)^{1/3}. \quad (20)$$

As soon as we write down this expression, students reach for their calculators – an opening for us to wax eloquent on the evils of calculators, and to show how to do the calculation by hand. We write 300 as $0.3 \cdot 10^3$; then the powers of ten cancel, leaving only $0.3^{1/3} \text{ m s}^{-1}$. So $v \sim 1 \text{ m s}^{-1}$,

which is 2 mph. A useful approximation: $1 \text{ m s}^{-1} \sim 2 \text{ mph}$. Is this speed reasonable? Yes – when I run in water, I cannot keep up with someone strolling alongside on the edge of the pool (a typical walking speed is 3 mph). The agreement with everyday experience increases our confidence in the drag formula. We can also point out that, even if we estimated the P_{avail} inaccurately (and we probably did), the error in the speed is small because of the blessed one-third power in the speed expression (20).

2.7 Drag force for a car

Emboldened, we use the drag formula for the original question, gas mileage. What is the frontal area for a car? A car is not as tall as a person, so the height is 1.5 m. When I go car camping and sleep in the back seat, I fit but do not consider it luxury accommodation; so the car’s width is maybe 1.5 m. The area is therefore $1.5 \text{ m} \times 1.5 \text{ m} \sim 2 \text{ m}^2$. To estimate the speed, pick a typical highway speed: 60 mph, or 30 m s^{-1} .

There are many ways to estimate the density of air. One method is to remember that 22 ℓ is one mole at standard conditions (sea-level pressure and room temperature). Air is mostly dinitrogen (N_2), with a molecular weight of 28. So 22 ℓ has a mass of 28 g. The density is roughly $1 \text{ g } \ell^{-1}$ or 1 kg m^{-3} . A more involved method derives the 22 ℓ magic number from the ideal gas law. For one mole, $PV = RT$, where P is pressure, V is the volume of one mole, R is the gas constant, and T is the temperature. We can look up the gas constant, and we know the temperature. Atmospheric pressure is easy to remember from (American) weather reports: ‘Barometer is 30 inches and falling’. One inch is 25 mm, so atmospheric pressure is equivalent to a column roughly 750 mm high. But 750 mm of what? Of mercury. Mercury is 13 times denser than water, so atmospheric pressure is equivalent to a column of water roughly $13 \times 750 \text{ mm}$ high, or $h \sim 10 \text{ m}$. The resulting pressure is, from hydrostatics,

$$P = \rho gh \sim 10^3 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2} \times 10 \text{ m} = 10^5 \text{ N m}^{-2}. \quad (21)$$

The volume occupied by one mole of atmosphere is

$$V = \frac{RT}{P} \sim \frac{8 \text{ J K}^{-1} \times 300 \text{ K}}{10^5 \text{ N m}^{-2}} \sim 24 \ell. \quad (22)$$

This calculation is another one that students can do mentally. The method is simple: Do the important parts first. So we first count the powers of 10. Rewrite ‘8’ as $0.8 \cdot 10^1$; then there are three powers of 10 in the numerator, and five in the denominator, which combine into 10^{-2} . The remaining factors are small and easy to handle mentally: 0.8×3 , or 2.4. So the volume is $2.4 \cdot 10^{-2} \text{ m}^3$ or 24 ℓ . This method of determining the molar volume, which starts with the ideal gas law, shows students how much they can estimate without looking up many quantities. Such estimation develops number sense and connects otherwise disparate bits of physics.

We now have computed the numbers that we need to estimate the drag force:

$$F \sim 1 \text{ kg m}^{-3} \times (30 \text{ m s}^{-1})^2 \times 2 \text{ m}^2 \sim 2 \cdot 10^3 \text{ N}. \quad (23)$$

This mental calculation is simple using the identity $30 \times 30 = 1000$. Other useful order-of-magnitude rules of arithmetic include

$$\begin{aligned} 2 \times 2 \times 2 &= 10, \\ 4 \times 4 &= 20, \\ \pi &= 3. \end{aligned} \quad (24)$$

3 Energy of gasoline

How much energy does a car get from 1 gallon of gasoline? What is this absurd unit, the gallon? It is 4 quarts; each quart is roughly one liter, so for our purposes, 1 gallon is 4ℓ. But 4ℓ of what? Gasoline is like fat in the energy that it stores. The nutrition information on the back of a soup can tells us that fat gives 10 calories per gram or $4 \cdot 10^4 \text{ J g}^{-1}$. [Let the students use that number (converting incorrectly to Joules at 4 cal J^{-1}) and complete the calculation. When they compute a horribly low mileage, ask why. Eventually students realize that nutritional calories are *kilocalories*. They can then redo the calculation using the proper conversion.] A favorite question: How reasonable is this value? To judge it, we should get a second opinion, for example from chemistry. Most chemical reactions release a few eV per molecule. For a long-chain hydrocarbon like gasoline, a molecular unit, say CH_2 , might be a better basis for that calculation. The molar mass of CH_2 is 14 g, so the energy density would be:

$$\frac{3 \text{ eV} \times 6 \cdot 10^{23}}{14 \text{ g}} \times \frac{1.6 \cdot 10^{-19} \text{ J}}{1 \text{ eV}} \sim 2 \cdot 10^4 \text{ J g}^{-1}.$$

Given the uncertainty in the pieces of this calculation, it agrees reasonably well with the soup-label estimate of $4 \cdot 10^4 \text{ J g}^{-1}$. What is the mass of 4ℓ of gasoline? In the order-of-magnitude world, every liquid is water, so 4ℓ has a mass of 4000 g. Its energy content is $4 \cdot 10^4 \text{ J g}^{-1}$, so the energy provided by 1 gallon is

$$E_{\text{gallon}} \sim 4000 \text{ g} \times 4 \cdot 10^4 \text{ J g}^{-1} \sim 2 \cdot 10^8 \text{ J}, \quad (25)$$

where the last step follows from the ‘identity’ $4 \times 4 = 20$.

4 Mileage

The energy that the car requires is the drag force times the distance traveled, d . Thus $E_{\text{avail}} = Fd$. The distance traveled is

$$d \sim \frac{E}{F} \sim \frac{2 \cdot 10^8 \text{ J}}{2 \cdot 10^3 \text{ N}} \sim 10^5 \text{ m}, \quad (26)$$

or 100 km. Our prediction – a 60 miles-per-gallon car – is reasonable.

We got a bit lucky. The drag is roughly one-fourth of what we estimated; the formula leaves out a factor of $0.5c_d$, where c_d is the drag coefficient (typically 0.5 for most cars – I once saw an ad for a sports car that quoted 0.33). The efficiency of the engine is not 1.0, but more like 0.25, which is also the efficiency of human metabolism. The two errors canceled, and we got an unreasonably accurate value. But that cancellation shows another advantage of order-of-magnitude methods: If you split the problem into enough parts, the errors in the different parts may cancel!

Our mileage estimate is reasonable, so we have answered our original question: Air resistance does cause a significant amount of the total resistance, at least at highway speeds. This analysis suggests a follow-up question: How much extra oil would the United States require if everyone drove 80 mph instead of 60 mph on the highway?

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6 References

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