# Ph 110a

*Assignment 8 soltions*

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## 1. Griffiths, 5.9

### ■ a)

Consider the following circuit: At $\rho = a$, $\phi$ goes from 0 to $\pi/2$. At $\phi=\pi/2$, $\rho$ goes from $a$ to $b$. At $\rho$=b, $\phi$ runs from $\pi/2$ to 0. At $\phi$=0, $\rho$ runs from $b$ to a. The circuit carries a uniform current $I$.

What is the magnetic field at the origin?

Well, the two places where the $\rho$ changes do not contribute (as $I$ points towards and away from the origin along those circuit segmants, so $\boldsymbol{dI} \times \hat{r} = 0$).

The contribution from the two quarter circles we simply calculate using Biot-Savart:

$B = \frac{\mu_0}{4\pi} I \int \frac{dl \times \hat{r}}{r^2}$

We take $dl = a\, d\phi\, \hat{\phi}$, over the range 0 to $\pi/2$ (for $\rho$=a), and $dl = -b\, d\phi\, \hat{\phi}$ (for $\rho$=b), over the same range. We note that $\boldsymbol{r} = -\boldsymbol{\rho}$. We also note that: $\hat{\phi} \times \hat{r} = \hat{\phi} \times (-\hat{\rho}) = \hat{\rho} \times \hat{\phi} = \hat{z}$. So we've got:

$B(0, 0, 0) = \frac{\mu_0}{4\pi} I \left( \int_{\phi'=0}^{\phi'=\pi/2} \left( \frac{\hat{z}\, a}{a^2} - \frac{\hat{z}\, b}{b^2} \right) d\phi \right) = \boxed{\frac{\mu_0}{8\pi} I \left( \frac{1}{a} - \frac{1}{b} \right) \hat{z}}$, where $\hat{z}$ is out of the page.

### ■ b

We've got current $I$ on a wire coming in from x=+∞ at y=-R, forming a half-circle around the origin of radius R (from $\phi=-\pi/2$ to $\phi=\pi/2$ going clockwise) then leaving at y=R off again to x=∞. What's the magnetic field at the origin?

We can break this into two bits: We recognize via symmetry that the magnetic field of two infinite line segments from x=0 to x=∞ at $y = \pm R$, is the same as one infinite wire (from x=∞ to x=-∞) with current going in the negative $\hat{x}$ direction on $y = -R$, or an infinite wire (from x=-∞ to x=∞) at $y = +R$. That would simply be $B_{\text{straight}} = \frac{\mu_0 I}{2\pi (s=R)} (-\hat{z})$ (from example 5.5)

The second bit is the half-circle. This is again a straightforward application of Biot-Savart similar to one of the arcs in part B. Using $\boldsymbol{dl} = -R\, d\phi\, \hat{\phi}$ over the range $\phi=\pi/2$ to $3\pi/2$, and recognizing that $-d\phi \times -\hat{\rho} = -\hat{z}$we calculate:

$B_{\text{arc}} = \frac{\mu_0}{4\pi} I \left( \int_{\phi'=\pi/2}^{\phi'=3\pi/2} \left( \frac{-\hat{z}\, R}{R^2} \right) d\phi \right) = \boxed{-\frac{\mu_0}{4R} I \hat{z}}$

So the total magnetic field is: $B = B_{\text{arc}} + B_{\text{straight}} = \boxed{\frac{\mu_0 I}{4R} \left( \frac{2}{\pi} + 1 \right) (-\hat{z})}$ (into the page)

## 2. Spinning Ball

Consider a uniformly charged spherical shell, of surface charge density $\sigma_0$ and radius $a$, rotating about the z-axis, with angular velocity $\omega$, which is taken to be located through the center of the sphere.

### ■ a)

What is the surface current density associated with this moving charge distribution?

Surface current density is going to be $\sigma_0\, v$, but $v$ is going to be dependent upon the the angle $\theta$: $v(\theta) = R\sin(\theta)\times\,\omega$, so we have: $\boxed{\boldsymbol{K} = \sigma\, R\sin(\theta)\,\omega\,\hat{\phi}}$

(b) What is the magnetic field found along the z-axis associated with these surface currents, both inside and outside of the shell? The professor would like us to do this problem by employing the Biot-Savart law.

#### Approach that works:

Let's try superimposing a bunch of hoops.

Each hoop has a current equal to:

$\boldsymbol{I} = (K)\,(R\,d\theta) = \sigma\,R^2\,\sin(\theta)\,\omega\,d\theta\;\hat{\phi}\;=\;\sigma\,R\,(R^2 - z'^2)^{1/2}\;\omega\,d\theta\,\hat{\phi}$, where z' is the height of the hoop.   Note: $d\theta = \frac{dz'}{\sqrt{R^2 - z'^2}} \Longrightarrow$
$I = \sigma\,\omega\,R\,dz\;\hat{\phi}$

We recall (Eqn 5.38 in the text found using Biot-Savart) that the magnetic field of a hoop at the origin of radius $r$ is:

$B(z) = \frac{\mu_0\,I}{2}\,\frac{r^2}{(r^2+z^2)^{3/2}}$. Let's consider a hoop located at some $z'$:

$B_{z'}(z) = \frac{\mu_0\,I}{2}\,\frac{r^2}{(r^2+(z-z')^2)^{3/2}}$, and let's consider the radius of the hoop as a function of z':

$r(z') = \sqrt{R^2 - z'^2}$ , and so:

$B_{z'}(z) = \frac{\mu_0\,2\,\sigma\,\omega\,R}{2}\,(R^2 - z')\,\frac{dz'}{(R^2+z^2-2\,z\,z')^{3/2}} = \frac{\mu_0\,\sigma\,\omega\,R}{2}\,\frac{(R^2-z'^2)}{(R^2+z^2-2\,z\,z')^{3/2}}\,dz'$

So we integrate from $z' = -R$ to $R$.   If $R > z$, then: $\int_{-R}^{R}\frac{(R^2-z'^2)}{(R^2+z^2-2\,z\,z')^{3/2}}\,dz' = \frac{4}{3}$,

if R<z then: $\int_{-R}^{R}\frac{(R^2-z'^2)}{(R^2+z^2-2\,z\,z')^{3/2}}\,dz' = \frac{4\,R^3}{3\,z^3}$. Giving us:

$B_{\text{inside}}(0,0,z) = \frac{2}{3}\,\mu_0\,\sigma\,\omega\,R$, and $B_{\text{outside}}(0,0,z) = \frac{2}{3}\,\mu_0\,\sigma\,\omega\,\frac{R^4}{z^3}$, which matches the solutions in the book.

#### Alternate approach (from "scratch")

Well, Biot Savart tells us: $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\text{surface}} K\,\frac{d\boldsymbol{a}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\int_{\text{surface}} K\,\frac{d\boldsymbol{a}'\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^{3/2}}$,

where K is the surface current, $d\mathbf{a}'$ is the differential area of our surface, and $\mathscr{r}$ is the vector pointing from the location of our source (the differential surface area) to $\boldsymbol{r}$.

Well a generic location would be $\boldsymbol{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$, but we're interested in only the field along the $z$ axis, so we restrict: $\boldsymbol{r} = z\,\hat{z}$.

Our location of differential surface current will be constrained to the sphere, and let's call it:
$r' = R\cos(\phi)\sin(\theta)\,\hat{x} + R\sin(\phi)\sin(\theta)\,\hat{y} + R\cos(\theta)\,\hat{z}$ .

So the vector pointing from our differential surface current to $\boldsymbol{r}$ will be:

$$\vec{\mathfrak{r}} = -R\cos(\phi)\sin(\theta)\,\hat{x} - R\sin(\phi)\sin(\theta)\,\hat{y} + (z - R\cos(\theta))\,\hat{z}$$

$$\mathfrak{r} = \sqrt{(R\cos(\phi)\sin(\theta))^2 + (R\sin(\phi)\sin(\theta))^2 + (z - R\cos(\theta))^2}$$
$$= \sqrt{R^2 + z^2 - 2Rz\cos(\theta)}$$

In our case: $d a' = R\,d\theta\,R\sin(\theta)\,d\phi\,\hat{\phi}$, where $\hat{\phi} = (\cos(\phi)\,\hat{y} - \hat{x}\sin(\phi))$.

So:

$$d a' \times \mathfrak{r} = R\,d\theta\,R\sin(\theta)\,d\phi(\cos(\phi)\,\hat{y} - \hat{x}\sin(\phi)) \times (-R\cos(\phi)\sin(\theta)\,\hat{x} - R\sin(\phi)\sin(\theta)\,\hat{y} + (z - R\cos(\theta))\,\hat{z})$$
$$= R^2\ \sin(\theta)\,d\theta\,d\phi \begin{pmatrix} z\cos(\phi)\sin(\theta) - R\cos(\theta)\cos(\phi)\sin(\theta) \\ z\sin(\theta)\sin(\phi) - R\cos(\theta)\sin(\theta)\sin(\phi) \\ R\cos^2(\phi)\sin^2(\theta) + R\sin^2(\phi)\sin^2(\theta) \end{pmatrix}$$
$$= R^2\ \sin(\theta)\,d\theta\,d\phi \begin{pmatrix} (z - R\cos(\theta))\cos(\phi) \\ (z - R\cos(\theta))\sin(\phi) \\ R\sin(\theta) \end{pmatrix}$$

So our B field looks like:

$$B(0, 0, z) = \frac{\mu_0}{4\pi}\int_{\text{surface}} K\,\frac{\boldsymbol{da}'\times\mathfrak{r}}{\mathfrak{r}^{3/2}}$$
$$= \frac{\mu_0}{4\pi}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \frac{(\sigma R\sin(\theta)\,\omega)}{(R^2 + z^2 - 2Rz\cos(\theta))^{3/2}}\,(R^2\ \sin(\theta)\,d\theta\,d\phi) \begin{pmatrix} (z - R\cos(\theta))\cos(\phi) \\ (z - R\cos(\theta))\sin(\phi) \\ R\sin(\theta) \end{pmatrix}$$

We see that the $\phi$ integral over $\cos(\phi)$ and $\sin(\phi)$ will cause the $x$ and y components to vanish,

as we could've reasoned from symmetry. So we have:

$$B_z = \frac{\mu_0\,\sigma\,\omega}{4\pi}\int_{\theta=0}^{\theta=\pi} d\theta\,\frac{R^4\ \sin(\theta)^3}{(R^2 + z^2 - 2Rz\cos(\theta))^{3/2}}\int_{\phi=0}^{\phi=2\pi} d\phi$$
$$= \frac{\mu_0\,\sigma\,\omega}{2}\int_{\theta=0}^{\theta=\pi} d\theta\,\frac{R^4\ \sin(\theta)^3}{(R^2 + z^2 - 2Rz\cos(\theta))^{3/2}}$$

That's a hell of an integral, no? But it is now convenient to go back to cartesian coordinates. We want to integrate over z' instead of $\theta$. We accomplish this by noting that $\sin(\theta) = \frac{\sqrt{R^2 - z'^2}}{R}$, and $R\cos(\theta) = z' \Longrightarrow d\theta = -\frac{dz'}{R\sin(\theta)} = \frac{dz'}{\sqrt{R^2 - z'^2}}$ We then have the integral:

$$B_z = \frac{\mu_0\,\sigma\,\omega}{2}\int_{z'=-R}^{z'=R} \frac{(R^2 - z'^2)^{3/2}}{(R^2 + z^2 - 2zz')^{3/2}}\ \frac{dz'}{\sqrt{R^2 - z'^2}} = \frac{\mu_0\,\sigma\,\omega}{2}\int_{z'=-R}^{z'=R} \frac{(R^2 - z'^2)}{(R^2 + z^2 - 2zz')^{3/2}}\,dz'$$

Which is the same integral as in the approach above. We're done.

## 3. More physical torroidal coil

In Griffiths, Example 5.10, the analysis of the toroidal coil presented is not a good physical model for an actual coil, which is wound from a single coil. This means that, in addition to the currents discussed in the example, there is an average current in the azimuthal (circumferential, $\phi$) direction, and the field is not purely azimuthal.

(a) What (simply) must this current be?

The current must be $I$. Consider the geometry of loops like the toroidal coil pictured in the book. Now at the top of each loop seperate them, and join one end to the loop to the left with and the other end of the loop to the loop to the right with circumferential segmants. The current that passes through each of those segmants must be $I$, because that's what's passing through each loop.

(b) Taking the radius s of this current to be approximated by it's average value, what is the magnetic field along the z-axis?

We can simply apply Example 5.6, treating the toroid as a loop of radius $\langle s \rangle$:

$$\boldsymbol{B}(z) = \frac{\mu_0 I}{2} \frac{\langle s \rangle^2}{(\langle s \rangle^2 + z^2)^{3/2}} \hat{z}$$

## 4. Two coaxial cylinders

Two infinitely long cylinders (with their center on the z-axis) of radii $a$, and b ($a < b$) carry uniformly distributed surface currents along the $z$-direction which have opposite and equal *total* values (+I and -I respectively).

a) Find the magnetic field everywhere in space.

We can exploit the symmetry of the sitution to apply Amperes law:

Taking an amperian loop of radius $s$ about the origin.

$$\oint B \cdot dl = B_\phi \oint dl = B_\phi \, 2\pi \mathrm{s} = \mu_0 I_{\mathrm{enc}}(s).$$

So we find:

$$B_\phi = \frac{\mu_0}{2\pi s} \times \begin{pmatrix} 0, \text{ if } s < a \\ I, \text{ if } a < s < b \\ 0, \text{ if } s > a \end{pmatrix}$$

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b) What is the magnetic force per unit area on each distribution?

Well, first let's note the surface currents:

$K_{\text{cylinder}} \; = \; \frac{I}{2 \pi R_{\text{cylinder}}}$, as the length perpindicular to the flow is the circumference. So: $K_{\text{inner}} \; = \; \frac{I}{2 \pi a}$, $K_{\text{outter}} \; = \; \frac{-I}{2 \pi b}$.

There's no magnetic field due to the outter cylinder inside it's radius, so there can be no magnetic force due to the outside cylinder. There is, however, a magnetic force on the outside cylinder due to the current carried by the inside cylinder:

$\frac{dF}{d\text{area}} = K_{\text{outter}} \times B_{\text{inner}} \; = \; \frac{-I}{2 \pi b} \, \hat{z} \times \frac{\mu_0}{2 \pi b} \, I \, \hat{\phi} = \frac{I^2}{4 \pi^2 b^2} \, \mu_0 \, \hat{s}$. So the outer cylinder feels a force outwards. If you find yourself frustrated due to the seeming violation of Newton's law, pat yourself on the back. We'll get into what's missing in our description in Chapter 7.

## 5. Constant field between overlapping cylinders

Consider a current distribution in which two cylinders of *uniform* longitudinal current (positive on the left, negative on the right, axes seperated in the $x-$ direction by a distance $d$), cancel in the region where they overlap. Show that the magnetic field in the overlap (current-free) region is constant.

Well, this is clear if $d = 0$. Let' $s$ explore other cases :

We approach this using superposition. Let us first consider one cylinder of current density $J_z$, and use Ampere's law with the amperian loop centered at the axis of the cylinder:

$I_{\text{enc}} \; = \int J \, d a = \int_0^{2\pi} d\phi \int_0^s d s \, J \, s = \pi \, s^2 \, J.$ So we have:

$B_\phi \times 2 \pi \, s \; = \; \pi \, s^2 \, J \Longrightarrow B_\phi \; = \; \frac{s J}{2}.$

So if the cylinder is centered at the origin: $B_{\text{cyl}} \; = \; -\frac{s J}{2} \sin(\phi) \, \hat{x} \; + \; \frac{s J}{2} \cos(\phi) \, \hat{y}$

$\; = \; \frac{J}{2} \, (-y \, \hat{x} \; + \; x \, \hat{y})$

Shifting to a cylinder off origin (located at x', y') would give us:

$B_{\text{cyl}}{}' = \frac{J}{2} \, (-(y - y') \, \hat{x} + (x - x') \, \hat{y})$

Now let us overlap one cylinder at the origin of current density $= J_z$, and one cylinder a distance $d$ away along the $x$ axis of current density $= -J_z$:

$B = \frac{J}{2} \, (-y \, \hat{x} + x \, \hat{y}) - \frac{J}{2} \, (-y \, \hat{x} + (x - d) \, \hat{y}) = \frac{J}{2} \, d \, \hat{y}$, which is constant.