Assignment 5
Ph 110b
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1 Griffiths, 9-19

■ a) Poor conductor

Show that the skin depth in a poor conductor ($\sigma << \omega\,\epsilon$) is $2/\sigma\,\sqrt{\epsilon/\mu}$ independent of frequency. Find the skin depth (in meters) for (pure) water.

We're given the definition of skin depth $d \equiv \frac{1}{\kappa}$, where $\kappa \equiv \omega\,\sqrt{\frac{\epsilon\,\mu}{2}}\,\left(\sqrt{1+(\frac{\sigma}{\epsilon\,\omega})^2}\,-1\right)^{1/2}$

We recall what it means to series expand $f(a)$ about small a: $f(a) \to f(0) + f'(0)\,a + \frac{1}{2}\,f''(0)\,a^2 + O(a^3)$

Well, if we series expand $\sqrt{1+a^2} \longrightarrow 1 + \frac{a^2}{2} + O(a^3)$.
So expanding $\sqrt{\sqrt{1+a^2}\,-1} \longrightarrow \frac{a}{\sqrt{2}} + O(a^3)$, or substituting in our original expression we have:

$\kappa = \frac{1}{2}\,\sqrt{\epsilon\,\mu}\,\sqrt{\frac{1}{\epsilon^2\,\omega^2}}\,\omega\,\sigma + O(\sigma^3) \,=\, \frac{1}{2}\,\sqrt{\frac{\mu}{\epsilon}}\,\sigma + O(\sigma^3)$

So to leading order in σ, we have $d = \frac{1}{\kappa} = \frac{2}{\sigma}\,\sqrt{\frac{\epsilon}{\mu}}$

For pure water we have $\epsilon = 80.1\,\epsilon_0$ (Table 4.2), $\mu = \mu_0(1+\chi_m)$,(Table 6.1)

$\sigma = 1/(2.5\times 10^5)$ Table (7.1), so $d = 1.19\times 10^4\,m$

■ b) Good conductor

Show that the skin depth in a good conductor ($\sigma >> \omega\,\epsilon$) is $\lambda/(2\,\pi)$ independent of frequency. Find the skin depth (in meters) for a typical metal ($\sigma = 10^7\,(\Omega\,m)^{-1}$) in the visible range ($\omega \simeq 10^{15}\,/s$), assuming $\epsilon = \epsilon_0$, and $\mu = \mu_0$

Again we have $\kappa = \omega\,\sqrt{\frac{\epsilon\,\mu}{2}}\,\left(\sqrt{1+(\frac{\sigma}{\epsilon\,\omega})^2}\,-1\right)^{1/2}$, except this time σ is large.

In this situation, we have $\frac{\sigma}{\epsilon\,\omega}$, dominating the square root, and so $\kappa \simeq \omega\,\sqrt{\frac{\epsilon\,\mu}{2}}\,\sqrt{\frac{\sigma}{\epsilon\,\omega}} = \sqrt{\frac{\mu\omega\sigma}{2}}$. Similarly

$k = \omega\,\sqrt{\frac{\epsilon\,\mu}{2}}\,\left(\sqrt{1+(\frac{\sigma}{\epsilon\,\omega})^2}\,+1\right)^{1/2} \simeq \sqrt{\frac{\mu\,\omega\,\sigma}{2}}$, is this regime so we have $\kappa \simeq k$. $k = \frac{2\,\pi}{\lambda} \Longrightarrow$

$d = \frac{1}{\kappa} \simeq \frac{1}{k} = \frac{\lambda}{2\,\pi}$.

So $\kappa = \sqrt{\frac{\mu\,\omega\,\sigma}{2}} = \sqrt{\frac{4\,\pi\times 10^{-7}\times 10^{15}\times 10^{7}}{2}} = 7.92665\times 10^7\,\frac{1}{\text{meter}}$, $d = \frac{1}{\text{K}} = 1.26\times 10^{-8}$ meters. Not very far at all.

c) *Lag*

Show that in a good conductor the magnetic field lags the electric field by $\pi/4$ radians, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part(b).

We get our lag from the phase given in eqn 9.134:

$\phi = \arctan(\kappa/k) = \arctan(1) = \pi/4$.

Eqn 9.137 gives us $\frac{B_0}{E_0} = \sqrt{\frac{\sigma\mu}{\omega}} = \sqrt{\frac{10^7\ 4\pi\times10^{-7}}{10^{15}}} = 1.121\times10^{-7}$.

2 Linear Approximation

$\partial_t\, \boldsymbol{J} = -\partial_t\, e\, n\, \boldsymbol{v}_e \simeq -e\, n_0\, \partial_t\, \boldsymbol{v} = -e\, n_0\, \boldsymbol{a} = -e\, n_0 \left(\frac{1}{m_e}\boldsymbol{F}\right) = -e\, n_0\left(\frac{-e}{m_e}\boldsymbol{E}\right) = +\frac{e^2\, n_0}{m_e}\boldsymbol{E}$.

Our wave equation is:

$\nabla^2\, \boldsymbol{E} = \mu\,\epsilon\,\partial_t{}^2\, \boldsymbol{E} + \mu\,\sigma\,\partial_t\, \boldsymbol{E} = \mu\,\epsilon\,\partial_t{}^2\, \boldsymbol{E} + \mu\,\partial_t\, \boldsymbol{J}$

$= \mu\,\epsilon\,\partial_t{}^2\, \boldsymbol{E} + \frac{\mu\, e^2\, n_0}{m_e}\boldsymbol{E}$

$\Longleftrightarrow$

$\left(\nabla^2 - \frac{\mu\, e^2\, n_0}{m_e} - \mu\,\epsilon\,\partial_t{}^2\right)\boldsymbol{E} = 0 \Longleftrightarrow$

$\left(\frac{1}{\mu\,\epsilon}\nabla^2 - \frac{e^2\, n_0}{\epsilon\, m_e} - \partial_t{}^2\right)\boldsymbol{E} = 0 \Longleftrightarrow$

$(c^2\,\nabla^2 - \omega_p{}^2 - \partial_t{}^2)\,\boldsymbol{E} = 0$, where $\omega_p{}^2 = \frac{e^2\, n_0}{\epsilon_0\, m_e}$.

b

Substitute $\boldsymbol{E} \propto \exp[i\,(k\,z - \omega\, t)]$ to find the dispersion relation $\omega(k)$ associated with this wave equation:

Plugging in we find: $-c^2\, e^{i\,(k\,z - t\,\omega)}\, k^2 - e^{i\,(k\,z - t\,\omega)}\,\omega_p^2 + e^{i\,(k\,z - t\,\omega)}\,\omega^2 = 0 \Longleftrightarrow$

$-c^2\, k^2 + \omega^2 - \omega_p^2 = 0$, or $\omega = \sqrt{c^2\, k^2 + \omega_p{}^2}$

c

For copper, the conduction electron density is approximately $n_0 \simeq 8.5\times10^{22}\ \text{cm}^{-3}$

Find ω_p. Well, using:

$e \to 1.60218\times10^{-19}$ Coulomb

$n_0 \to \frac{8.5\times10^{22}}{\text{Meter}^3}$

$\epsilon_0 \to \frac{8.85419\times10^{-12}\text{ Ampere Second}}{\text{Meter Volt}}$

$m_e \to 9.10938\times10^{-31}$ Kilogram

we find:

$\omega_p{}^2 = \frac{e^2\, n_0}{\epsilon_0\, m_e} = 2.70522\times10^{32}\ \frac{1}{\text{second}^2} \Longrightarrow$

$\omega_p = 1.64475\times10^{16}\ \frac{1}{\text{second}}$

■ d

Discuss the attentuation of an electromagnetic wave with $\lambda = 700$ nm impinging on a copper mirror.

My original (sloppy) interpretation:

Well, $\lambda = \frac{2\pi}{k}$, and for a good conductor we have: $k \simeq \kappa$, so $\kappa \simeq \frac{2\pi}{\lambda}$, and the wave attenuates as: $e^{-\kappa z} = e^{\frac{-2\pi z}{7}\times10^7}$, if z is given in meters. So it drops off pretty quickly. (i.e. it reaches half of its initial amplitude in: 7.72225×10^{-8} meters of material)

What the professor is looking for:

The professor's wants us to explore the attenuation due to propagation below cutoff in a plasma, $\omega<\omega_p$, (λ_p is 114 nm, and λ is 700 nm). So the attentuation constant is given by $\kappa = \sqrt{\omega_p{}^2 - \omega^2} \Big/ c = 5.48252\times10^7\ m^{-1}$

in contrast to simply $\frac{2\pi}{\lambda} = 8.97598\times10^6\ m^{-1}$, which is off by an order of magnitude.

■ e

What is the phase velocity of a soft X-ray with $\lambda = 5\times10^{-9}\ m$ in copper? v_g?

$\omega = \sqrt{c^2\, k^2 + \omega_p{}^2} \Longrightarrow$

$\frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p{}^2}{k^2}} = c\sqrt{1 + \frac{\omega_p{}^2}{k^2\, c^2}}$

$= c\sqrt{1 + \frac{2.70522\times10^{32}}{(2\pi/5\times10^{-9})^2\, 8.98755\times10^{16}}}\ m/s$

$= 1.00095\, c$

We calculate the group velocity:

$\frac{d\omega}{dk} = \frac{c^2\, k}{\sqrt{c^2\, k^2 + \omega_p^2}} = c \times \frac{k}{\sqrt{k^2 + \frac{\omega_p{}^2}{c^2}}} = 0.999048\, c$

scratch

3 Griffiths, 9.29

Confirm that the energy in the TE_{mn} mode travels at the group velocity.

Hint: Find the time averaged Poynting vector ($\langle S\rangle$) and the energy density $\langle \mu\rangle$ (use Prob 9.11).

Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave and take their ratio.

Well, from problem 9.11, we see that:

$\langle u\rangle = \frac{1}{4}\left(\epsilon_0\, \tilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{E}}^* + \frac{1}{\mu_0}\tilde{\boldsymbol{B}}\cdot\tilde{\boldsymbol{B}}^*\right)$ and $\langle S\rangle = \frac{1}{2\mu_0}\left(\tilde{\boldsymbol{E}}\times\tilde{\boldsymbol{B}}^*\right)$

As we're going to be integrating over the cross section the only relevant component from $\langle \boldsymbol{S}\rangle$ is $\langle S\rangle_z$.

What does a wave in a waveguide look like? It looks like 9.180:

$$E_x = \frac{i}{(\omega/c)^2 - k^2}(k\,\partial_x E_z + \omega\,\partial_y B_z)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}(k\,\partial_y E_z - \omega\,\partial_x B_z)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}(k\,\partial_x B_z - \frac{\omega}{c^2}\,\partial_y E_z)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}(k\,\partial_y B_z + \frac{\omega}{c^2}\,\partial_x E_z)$$

Ahh, but what about a TE_{mn} mode? TE means $E_z = 0$, and we're given

$B_z = B_0\cos(m\pi x/a)\cos(n\pi y/a)$ from equation 9.186. Plugging in:

$$E_x = \frac{i\omega}{(\omega/c)^2 - k^2}(\frac{-n\pi}{b} B_0 \cos(m\pi x/a)\sin(n\pi y/b))$$

$$E_y = \frac{-i\omega}{(\omega/c)^2 - k^2}(\frac{-m\pi}{a} B_0 \sin(m\pi x/a)\cos(n\pi y/b))$$

$$B_x = \frac{i k}{(\omega/c)^2 - k^2}(\frac{-m\pi}{a} B_0 \sin(m\pi x/a)\cos(n\pi y/b))$$

$$B_y = \frac{i k}{(\omega/c)^2 - k^2}(\frac{-n\pi}{b} B_0 \cos(m\pi x/a)\sin(n\pi y/b))$$

The rest is simply algebra. Just realize that the integral is over $d\boldsymbol{a}$, whose direction is $\hat{z}$.

$$\langle S\rangle_z = \frac{1}{2\mu_0}\epsilon_{zij}\,\tilde{E}_i\,\tilde{B}_j^* = \frac{1}{2\mu_0}\left(\tilde{E}_x\,\tilde{B}_y^* - \tilde{E}_y\,\tilde{B}_x^*\right)$$
$$= \frac{B_0^2\,\pi^2}{2\mu_0}\left(\frac{\omega k}{((\omega/c)^2 - k^2)^2}\right)\left((\frac{n}{b})^2\cos(m\pi x/a)^2\sin(n\pi y/a)^2 + (\frac{m}{a})^2\sin(m\pi x/a)^2\cos(n\pi y/b)^2\right)$$

Integrating we find:

$$\int\langle S\rangle_z\, da = \frac{B_0^2\,\pi^2}{2\mu_0}\left(\frac{\omega k}{((\omega/c)^2 - k^2)^2}\right)((\frac{n}{b})^2\frac{ab}{4} + (\frac{m}{a})^2\frac{ab}{4})$$

Now to calculate $\langle u\rangle = \frac{1}{4}\left(\epsilon_0\,\tilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{E}}^* + \frac{1}{\mu_0}\tilde{\boldsymbol{B}}\cdot\tilde{\boldsymbol{B}}^*\right)$:

$$\langle \mu \rangle = \frac{1}{4}\left(\epsilon_0 \frac{\omega^2 \pi^2}{((\omega/c)^2 - k^2)^2} B_0{}^2\left(\left(\frac{n}{b}\right)^2 \cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2 \sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi x}{b}\right)\right) + \right.$$
$$\frac{1}{\mu_0} \frac{k^2 \pi^2}{((\omega/c)^2 - k^2)^2} B_0{}^2\left(\left(\frac{m}{a}\right)^2 \sin^2(m\pi x/a)\cos^2(n\pi y/b) + \left(\frac{n}{b}\right)^2 \cos^2(m\pi x/a)\sin^2(n\pi y/b)\right) +$$
$$\left.\frac{1}{\mu_0} B_0{}^2 \cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi x}{b}\right)\right)$$

Integrating we find:

$$\int \langle u \rangle \, da = \frac{a\,b}{16}\left\{\epsilon_0 \frac{\omega^2 \pi^2}{((\omega/c)^2 - k^2)^2} B_0{}^2\left(\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right) + \frac{1}{\mu_0} \frac{k^2 \pi^2}{((\omega/c)^2 - k^2)^2}\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right) + \frac{B_0{}^2}{\mu_0}\right\}$$

We can use $\left(\left(\frac{\omega}{c}\right)^2 - k^2\right) = (\omega_{mn}/c)^2$, $\frac{1}{c^2 \mu_0} = \epsilon_0$, and $\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right) = \left(\frac{\omega_{mn}}{\pi c}\right)^2$, to clean these up a bit:

$$\int \langle S \rangle_z \, da = \frac{B_0{}^2 \pi^2}{2\mu_0}\left(\frac{\omega k}{((\omega/c)^2 - k^2)^2}\right)\left(\left(\frac{n}{b}\right)^2 \frac{a\,b}{4} + \left(\frac{m}{a}\right)^2 \frac{a\,b}{4}\right) = \frac{B_0{}^2 \pi^2}{2\mu_0}\left(\frac{\omega k}{(\omega_{mn}/c)^4}\right)\left(\frac{\omega_{mn}}{\pi c}\right)^2 \frac{ab}{4}$$
$$= \frac{B_0{}^2}{8\mu_0} \frac{a\,b\,c^2\,\omega k}{\omega_{mn}{}^2}$$

and (using $(k\,c)^2 = (\omega)^2 - (\omega_{mn})^2$)

$$\int \langle u \rangle \, da = \frac{a\,b}{16}\left\{\frac{1}{c^2 \mu_0} \frac{\omega^2 \pi^2}{((\omega_{mn}/c)^2)^2} B_0{}^2\left(\left(\frac{\omega_{mn}}{\pi c}\right)^2\right) + \frac{1}{\mu_0} \frac{k^2 \pi^2}{((\omega_{mn}/c)^2)^2}\left(\left(\frac{\omega_{mn}}{\pi c}\right)^2\right) + \frac{B_0{}^2}{\mu_0}\right\}$$
$$= \frac{a\,b}{16} \frac{B_0{}^2}{\mu_0}\left(\frac{\omega^2}{(\omega_{mn})^2} + \frac{k^2 c^2}{(\omega_{mn})^2} + 1\right)$$
$$= \frac{a\,b}{16} \frac{B_0{}^2}{\mu_0 (\omega_{mn})^2}\left(\omega^2 + (\omega)^2 - (\omega_{mn})^2 + \omega_{mn}{}^2\right) = \frac{a\,b\,B_0{}^2\,\omega^2}{8\mu_0(\omega_{mn})^2}$$

Great. So now we've got:

$$\frac{\int \langle S \rangle_z \, da}{\int \langle u \rangle \, da} = \frac{B_0{}^2}{8\mu_0} \frac{a\,b\,c^2\,\omega k}{\omega_{mn}{}^2} \Big/ \frac{a\,b\,B_0{}^2\,\omega^2}{8\mu_0(\omega_{mn})^2} = \frac{c^2 k}{\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}{}^2} = v_g$$

4 TEM Waves

In section 9.5.3, it is shown that the TEM wave travels with both phase and group velocity *c* in a coaxial conductor system.

■ a

Show that the electric and magnetic fields in the coaxial system arise from a charge per unit length and current that are related by *c*.

We're given that $E_s = \frac{A}{s}\cos(k\,z - \omega\,t)$, and $B_\phi = \frac{A}{c\,s}\cos(k\,z - \omega\,t)$

We can use Gauss's law to find the charge enclosed, specifically for a cylinder of radius *s* and length dz:

$$\oint E \cdot \mathrm{da} = \frac{Q_{enc}}{\epsilon_0} \Longleftrightarrow 2\pi s \times E_s\,dz = \frac{1}{\epsilon_0}\lambda\,d_z \Longrightarrow \lambda = \epsilon_0\,2\pi s\,E_s = \epsilon_0\,2\pi A\cos(k\,z - \omega\,t)$$

We can use Ampere's law to find the current enclosed, specifically for a circle of radius s:

$$\oint B \cdot dl = \mu_0 I_{enc} \Longrightarrow 2\pi s\,B_\phi = \mu_0 I_{enc} \Longrightarrow I = \frac{1}{\mu_0}\frac{2\pi}{c} A\cos(k\,z - \omega\,t) = \frac{\lambda}{c\mu_0\,\epsilon_0} = c\,\lambda$$

■ b

What is the phase and group velocity in the case that the conductors have a dielectric of frequency independent permittivity ε between them?

We're really playing a two dimensional game here, and we can simply say that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, and so we shift c to $\frac{1}{\sqrt{\mu_0 \epsilon}}$ in the dispersion relation: $k = \sqrt{\mu_0 \epsilon}\, \omega$. In this case $v = \frac{1}{\sqrt{\mu_0 \epsilon}}$, as does v_g.

If one actually takes the cartesian laplacian on the solutions above we find that the dispersion relation is really that simple. i.e. These solutions have to satisfy $\nabla^2 E_i = \mu_0 \epsilon \partial_t{}^2 E_i$. But for the TEM modes as given above (only x and y components when you translate to cartesian), we see that only the z derivative survives. i.e.:

$$E_x = E_s \cos(\phi) = E_s \frac{x}{s} = \frac{A x}{(x^2+y^2)} \cos(k z - \omega t)$$

Note: $(\partial_x{}^2 + \partial_y{}^2) \frac{x}{(x^2+y^2)} = \left(\frac{8 x^2}{(x^2+y^2)^3} - \frac{2}{(x^2+y^2)^2}\right) x + \left(\frac{8 y^2}{(x^2+y^2)^3} - \frac{2}{(x^2+y^2)^2}\right) x - \frac{4 x}{(x^2+y^2)^2}$

$$= \frac{8 x^3}{x^6+3 y^2 x^4+3 y^4 x^2+y^6} - \frac{8 x}{x^4+2 y^2 x^2+y^4} + \frac{8 y^2 x}{x^6+3 y^2 x^4+3 y^4 x^2+y^6}$$

$$= 0$$

So we find: $\partial_z{}^2 E_x = \mu_0 \epsilon \partial_t{}^2 E_x \Longleftrightarrow k^2 = \mu_0 \epsilon \omega^2$ which is the simple dispersion relation above.

The same holds for E_y, B_x, and B_y.

■ c

What is the relationship between the voltage between the two conductors and the current I?

$$E_s = \frac{\lambda}{\epsilon_0 2 \pi s} \Longrightarrow$$

$$\Delta V = -\int_a^b E_s \, ds = \frac{\lambda}{\epsilon 2 \pi} \ln\left(\frac{a}{b}\right) = \frac{\sqrt{\epsilon \mu_0}\; I}{\epsilon 2 \pi} \ln\left(\frac{a}{b}\right) = \sqrt{\frac{\epsilon \mu_0}{\epsilon^2}}\; \frac{I}{2\pi} \ln\left(\frac{a}{b}\right)$$

So the line impedance $Z = \frac{\Delta V}{I} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon}} \ln\left(\frac{a}{b}\right)$.

For a dielectric loaded coaxial guide with $\varepsilon = 3$, can we pick a ratio a/b to make this impedance 50 Ω?

Well, the ratio has to be:

$$\frac{a}{b} = e^{2 \pi z \sqrt{\frac{\epsilon}{\mu_0}}}$$

If we really mean $\varepsilon = 3 \times \epsilon_0$, then: $\frac{a}{b} = 4.2392$.

If $\varepsilon \equiv \chi_e$, then $\varepsilon = (1 + \chi_e) = 4 \times \epsilon_0$ and $\frac{a}{b} = 5.3006$

And if ε=3 $\frac{\text{Ampere Second}}{\text{Meter Volt}}$ (i.e. 3 × SI units for permittivity), then $\frac{a}{b} = 2.264309697 \times 10^{210809}$, which is unlikely.

5 Resonant Cavities

Consider the resonant cavity produced by closing off the two ends of a rectangular wave guide, at $z = 0$, and $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by:
$\omega_{\text{lmn}} = c\pi \sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$

Let's look at the boundary conditions: $E^{||} = 0$, and $B^{\text{perp}} = 0$. Also, as we've no stuff inside,

$\nabla \cdot E = 0$.

$E^{||} = 0$, means that:

$$E_y(a, y, z) = E_y(0, y, z) = E_z(a, y, z) = E_z(0, y, z) = 0$$

$$E_x(x, b, z) = E_x(x, 0, z) = E_z(x, b, z) = E_z(x, 0, z) = 0$$

$$E_x(x, y, d) = E_x(x, y, 0) = E_y(x, y, d) = E_y(x, y, 0) = 0$$

Maxwell's equations: $\nabla \times E = i\,\omega\,B$, $\nabla \times B = \frac{-i\,\omega}{c^2}\,E \Longrightarrow$

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = \nabla \times (i\,\omega\,B) = \frac{\omega^2}{c^2}\,E$$

Note: we need only consider the E field as the B field can be arrived at through $\nabla \times E$.

Our differential equation is: $\nabla^2 E_i = -\frac{\omega^2}{c^2}\,E_i$

For example: $\nabla^2 E_x = -\frac{\omega^2}{c^2}\,E_x$. Let: $E_x(x, y, z) = X(x)\,Y(y)\,Z(z)$. Separation of variables suggests, $\partial_i{}^2\,R_i(r_i) = -k_i{}^2\,R_i(r_i)$, for each component. Like so:

$$(\partial_x{}^2 + \partial_y{}^2 + \partial_z{}^2)\,E_x = \frac{-\omega^2}{c^2}\,E_x \Longrightarrow (\partial_x{}^2 + \partial_y{}^2 + \partial_z{}^2)\,X(x)\,Y(y)\,Z(z) = \frac{-\omega^2}{c^2}\,X(x)\,Y(y)\,Z(z)$$

$$\Longrightarrow (\partial_x{}^2\,X(x)\,Y(y)\,Z(z) + \partial_y{}^2\,X(x)\,Y(y)\,Z(z) + \partial_z{}^2\,X(x)\,Y(y)\,Z(z)) = \frac{-\omega^2}{c^2}\,X(x)\,Y(y)\,Z(z)$$

$$\Longrightarrow (\partial_x{}^2\,X(x))\,Y(y)\,Z(z) + X(x)\,Z(z)\,(\partial_y{}^2\,Y(y)) + X(x)\,Y(y)\,\partial_z{}^2\,(Z(z)) = \frac{-\omega^2}{c^2}\,X(x)\,Y(y)\,Z(z)$$

Now divide both sides by E_x:
$$\Longrightarrow \frac{(\partial_x{}^2\,X(x))}{X(x)} + \frac{(\partial_y{}^2\,Y(y))}{Y(y)} + \frac{\partial_z{}^2\,(Z(z))}{Z(z)} = -\frac{\omega^2}{c^2}$$

So now, we let each independent expression equal a constant (eg. $\frac{(\partial_x{}^2\,X(x))}{X(x)} = -k_x{}^2$) such that the sum of the constants is $-\omega^2/c$:

$$-k_x{}^2 - k_y{}^2 - k_z{}^2 = -\frac{\omega^2}{c^2}$$

So now we have the following differential equations:

$$\partial_x{}^2\,X(x) = -k_x{}^2\,X(x),\ \ \partial_y{}^2\,Y(y) = -k_y{}^2\,Y(y),\ \ \partial_z{}^2\,Z(z) = -k_z{}^2\,Z(z)$$

These solutions are sinusoids, and our parallel boundary condition removes cosine terms from the components parallel to the appropriate boundaries:

i.e we get something of the form:

$E = A\cos(k_x\, x)\sin(k_y\, y)\sin(k_z\, z)\,\hat{x} + B\sin(k_x\, x)\cos(k_y\, y)\sin(k_z\, z)\,\hat{y} + C\sin(k_x\, x)\sin(k_y\, y)\cos(k_z\, z)\,\hat{z}$

Where $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, and $k_z = \frac{l\pi z}{d}$ are required for the sin terms to vanish appropriately.

Note: only cosine terms survive on component (i.e. we get a $\cos(k_z\, z)$ for the $\hat{z}$ component, a $\cos(k_x\, x)$ for the $\hat{x}$ component, etc) due to the requirement of $\nabla\cdot E = 0$. Why? $\partial_z \cos(k_z\, z) \propto \sin(k_z\, z)$, which vanishes at the boundary.

As the sum over ${k_i}^2$ must equal $(\frac{\omega}{c})^2$ from our separation of variables, we have $\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2} = \frac{\omega_{\text{mnl}}}{c^2}$, and we're done.

6 Midterm problem

Consider a circular current loop located in the $z = 0$ plane, tracing out a radius a about the origin, and sinusoidal current excitation $I = I_0 \sin(\omega\, t)$.

■ a)

What is the time dependent magnetic dipole moment due to this current loop?

$m = I \times \text{area}^2 = I_0 \sin(\omega\, t)\,\pi\, a^2\,\hat{z}$

■ b)

Given the magnetic vector potential find the electric field.

Well: $\nabla \times E = -\partial_t B = -\partial_t (\nabla \times A) = -\nabla \times \partial_t A_t$

So modulo functions mapping to zero under the curl (i.e. gradients of scalar fields etc) :

$$
\begin{aligned}
\boldsymbol{E} &= -\partial_t A(t) \\
&= -\frac{\mu_0}{4}\, a^2\, \frac{I_0\ \omega\cos(\omega\, t)}{r^3}\left(\hat{z}\times\vec{r} = r\sin(\theta)\,\hat{\phi}\right) \\
&= -\frac{\mu_0}{4}\, a^2\, I_0\ \omega\cos(\omega\, t)\,\frac{s}{r^3}\,\hat{\phi}
\end{aligned}
$$