Assignment 6
Ph 110b
TA: John Joseph M. Carrasco
jjmc@physics.ucla.edu

1. Griffiths, 10-3.

Find the fields, and the charge and current distributions, corresponding to:

V(r,t)=0, $\boldsymbol{A}(\boldsymbol{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{q\,t}{r^2}\,\hat{r}$

There's very little to do actually, it's pretty straightforward calculation.

$\boldsymbol{E} = -\nabla V - \partial_t\, \boldsymbol{A} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\,\hat{r}$

$\boldsymbol{B} = \nabla \times \boldsymbol{A} = 0$

$\frac{1}{\epsilon_0}\,\rho = \nabla \cdot E = q\,\delta^3(\vec{r})$

$J \propto \partial_t\, E = 0$

It turns out this is just a funny way of describing a stationary point charge at the origin.

2. Griffiths, 10-5.

Use the gauge function $\lambda = -\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{q\,t}{r}\right)$ to transform the potential in Prob 10.3 and comment on the result.

$A' = A + \nabla\lambda = -\frac{1}{4\pi\epsilon_0} \frac{q\,t}{r^2} - \left(\frac{1}{4\pi\epsilon_0}\right)(-1)\,\frac{q\,t}{r^2} = 0$

$V' = V - \frac{\partial\lambda}{\partial t} = +\frac{q}{4\pi\epsilon_0\,r}$

The fields are, of course, unchanged.

Discuss the gauge choices made:

10.3:

This choice of gauge represents neither a Coulomb or Lorentz gauge.

10.5:

$\nabla \cdot A = 0 = \frac{\partial V}{\partial t}$ which satisfies both Coulomb and Lorentz gauges.

3. Faster then light

Consider a charged particle traveling in a straight line along the z-axis with velocity $v = .99\,c$, in a medium with index of refraction $n = 1.5$. since light travels with speed c/n in this medium, the retarded time that goes into potential calculations is $t_r = t - n\,r/c$, where r is the usual separation between the observation and particle positions. Show that charge contributes to the potentials on a cone traveling with the particle, as shown below. What is the angle of this cone as a function of arbitrary v and n?

$$V(r,\,t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r',t_r)}{|r-r'|}\,d r = \frac{1}{4\pi\epsilon_0}\,2\pi\int d z'\,\frac{\rho(r',t_r)}{\sqrt{(z-z')^2+(s)^2}}$$

But what is the density? $\rho(z,\,t) = q\,\delta(z - v\,t)$, so we have:

$$V(r,\,t) = \frac{q}{4\pi\epsilon_0}\int d z'\,\frac{\delta(z'-v\,t_r)}{\sqrt{(z-z')^2+(s)^2}}$$

Let's consider the integrand. It's zero whenever $z' - v\,t_r$ fails to vanish.

It's non-zero for:

$$z' - v\,t_r = 0$$

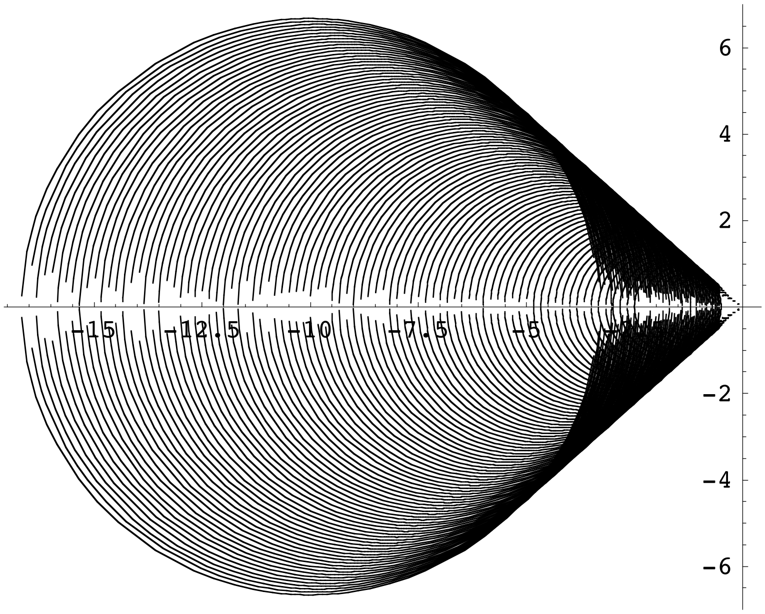
$$z' - v\,t + \sqrt{s^2 + (z - z')^2}\;n/c = 0$$

$$v\,t - z' = +\sqrt{s^2 + (z - z')^2}\;n/c$$

Note that for the radical to be real, we need $z' < v\,t$. Let's define the position of the charged particle as $z_q(t) = v\,t$. If we square both sides we recognize that we have the equation of a circle in z and s of radius: $(z_q - z')^2\,\frac{c^2}{n^2}$, centered at $s = 0$, $z = z'$:

$$(z_q - z')^2\,\frac{c^2}{n^2} = s^2 + (z - z')^2$$

We have such a circle for each point $z' < v\,t$. If we superimpose a bunch of these circles for $z' < z_q$, we see the emerging cone (rotating through ϕ about z) of non-zero potential:



What's the angle of the cone? The distance travelled by light in the medium in a fixed time t is: $\frac{c}{n}\,t$. The distance travelled by the particle in said time t, is: $v\,t$. We realize that any point tangent to a circle makes an angle of $\pi/2$ with the radial vector of said circle. This means that $\sin\theta$, is the ratio between the speed at which information is propagating from the particle, and the speed at which the particle is moving, so we have:

$$\sin(\theta) = \frac{c/n}{v} = \frac{c}{n\,v}.$$

scratch

4. Griffiths 10.14, More moving charge

Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.42) can be written equivalently as:

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2(\theta)}}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{v}\,\boldsymbol{t}$ is the vector from the present position of the particle to the field point $\boldsymbol{r}$, and θ is the angle between $\boldsymbol{R}$ and $\boldsymbol{v}$.

We start with the scalar potential given in 10.42:

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\,\frac{q\,c}{\sqrt{(c^2\,t-\boldsymbol{r}\cdot\boldsymbol{v})^2+(c^2-v^2)\,(r^2-c^2\,t^2)}}$$

This problem is pretty much just straightforward algebra. We want to remove all r's and t's from our expression in favour of R and θ, where θ is the angle between R and v.

$\boldsymbol{v}\,t = \boldsymbol{r} - \boldsymbol{R}$, and $\boldsymbol{R}\cdot\boldsymbol{v} = R\,v\cos(\theta)$, so $\boldsymbol{r} = \boldsymbol{v}\,t + R$, and $r^2 = r\cdot r = v^2\,t^2 + R^2 + 2\,R\,v\cos(\theta)\,t$, and $\boldsymbol{r}\cdot\boldsymbol{v} = (R\,v\cos(\theta) + v^2\,t)$

We consider the expression in the radical (the radicand if you like) in the denominator of the potential expression, and remove all uses of r, via the expressions above. Basically we plug in for r, and jiggle until our end expression plunks out:

$$
\begin{aligned}
&(c^2\, t - \boldsymbol{r}\cdot\boldsymbol{v})^2 + (c^2 - v^2)\,(r^2 - c^2\, t^2) = (t\, c^2 - t\, v^2 - R\, v\cos(\theta))^2 + (c^2 - v^2)\,(R^2 + 2\, t\, v\cos(\theta)\, R - c^2\, t^2 + t^2\, v^2) \\
&= t^2\, c^4 - 2\, t^2\, v^2\, c^2 - 2\, R\, t\, v\cos(\theta)\, c^2 + t^2\, v^4 + R^2\, v^2\cos^2(\theta) + \\
&\quad 2\, R\, t\, v^3\cos(\theta) + -t^2\, c^4 + R^2\, c^2 + 2\, t^2\, v^2\, c^2 + 2\, R\, t\, v\cos(\theta)\, c^2 - t^2\, v^4 - R^2\, v^2 - 2\, R\, t\, v^3\cos(\theta) \\
&= c^2\, R^2 - v^2\, R^2 + v^2\cos^2(\theta)\, R^2 \\
&= c^2\, R^2 - v^2\, R^2(1 - \cos^2(\theta)) \\
&= c^2\, R^2 - v^2\, R^2\sin^2(\theta)
\end{aligned}
$$

Substituting that back in to our potential expression:

$$V(\boldsymbol{r},\, t) = \frac{1}{4\,\pi\,\epsilon_0}\;\frac{q\, c}{\sqrt{c^2\, R^2 - v^2\, R^2\sin^2(\theta)}} = \frac{1}{4\,\pi\,\epsilon_0}\;\frac{q\Big/\sqrt{1 - \frac{v^2}{c^2}\sin^2(\theta)}}{R}$$

5. Alternate form for moving potential

There are many ways to approach the problem of the potentials associated with a moving point charge. Let us propose another: consider a charged particle traveling in a straight line along the $z-$axis with velocity v. Use the algebraic (not Lorentz, we must wait a week or so for that!) transformation of variables in the Inhomogeneous wave equation.

So basically we're shifting to the reference frame of the particle:

$\tilde{x} = x,\ \tilde{y} = y,\ \xi = z - v\, t$. We note that $\partial_{\tilde{x}} = \partial_x,\ \partial_{\tilde{y}} = \partial_y, \partial_z = \partial_\zeta$, and $\partial_t = -v\,\partial_\zeta$

So $\Box^2\, V = (\nabla^2 - \frac{1}{c^2}\,\partial_t{}^2)\, V = \left(\partial_{\tilde{x}}{}^2 + \partial_{\tilde{y}}{}^2 + \left(1 - (\frac{v}{c})^2\right)\partial_\zeta{}^2\right) V = \frac{-\rho_{f(\tilde{x},\tilde{y},\zeta)}}{\epsilon_0}$

We now rescale the longitudinal variable $\tilde{z} = \zeta\gamma$, where $\gamma = \left(1 - (\frac{v}{c})^2\right)^{-1/2}$:

$\partial_{\tilde{z}} = \frac{1}{\gamma}\,\partial_\zeta$, so $\partial_{\tilde{z}}{}^2 = \frac{1}{\gamma^2}\,\partial_\zeta{}^2$ and we have: $\tilde{\nabla}^2\, V = -\frac{\rho_f\left(\tilde{x},\tilde{y},\frac{\tilde{z}}{\gamma}\right)}{\epsilon_0}$.

$\rho_f\left(\tilde{x},\, \tilde{y},\, \frac{\tilde{z}}{\gamma}\right) = q\,\delta(\tilde{x})\,\delta(\tilde{y})\,\delta(\tilde{z}/\gamma) = q\,\gamma\,\delta(\tilde{x})\,\delta(\tilde{y})\,\delta(\tilde{z})$, so we have:

$\tilde{\nabla}^2\, V = -\gamma\,\frac{\rho_f(\tilde{x},\tilde{y},\tilde{z})}{\epsilon_0}$, and we're almost done.

So our differential equation is $\tilde{\nabla}^2\, V = -\gamma\,\frac{q\,\delta^3(\tilde{\boldsymbol{r}})}{\epsilon_0}$, but we've solved this before in electrostatics:

$$V(\tilde{\boldsymbol{r}}) = -\frac{1}{4\,\pi\,\epsilon_0}\,\frac{(\gamma\, q)}{\tilde{r}} = -\frac{1}{4\,\pi\,\epsilon_0}\left(\frac{q\Big/\sqrt{1 - (\frac{v}{c})^2}}{\tilde{r}}\right)$$

Comparing to the answer given in Problem 4, we find it's of a very similar form. What's the distinction? In problem 4, we're in the lab frame---the particle has some velocity v in a given direction, and the potential has to reflect that assymetry. In this problem we've solved for the potential in the particle's frame, which has no preferred direction.

6. Griffiths, 11.8

Apply Eqns 11.59 ($\boldsymbol{S} \simeq \frac{1}{\mu_0} (\boldsymbol{E} \times \boldsymbol{B}) = \frac{\mu_0}{16 \pi^2 c} [\ddot{p}(t_0)]^2 \left(\frac{\sin^2(\theta)}{r^2}\right) \hat{r}$) and 11.60 ($P \cong \int \boldsymbol{S} \cdot d\boldsymbol{a} = \frac{\mu_0 \ddot{p}^2}{6 \pi c}$) to the rotating dipole of Problem 11.4. Explain any apparent discrepancies with the previous answer.

$$\boldsymbol{p}(t) = p_0(\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}) \Longrightarrow$$

$$\ddot{\boldsymbol{p}}(t) = -\omega^2\, p_0\, (\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}) \Longrightarrow$$

$$\ddot{p}^2 = \omega^4\, p_0{}^2 (\cos^2(\omega t) + \sin^2(\omega t))$$
$$= \omega^4\, p_0{}^2$$

So $\boldsymbol{S} = \frac{\mu_0\, p_0{}^2\, \omega^4}{16 \pi^2 c} \left(\frac{\sin^2 \theta}{r^2}\right) \hat{r}$ and $P = \frac{\mu_0\, p_0{}^2\, \omega^4}{6 \pi c}$.

6. Griffiths, 11.14

In Bohr's theory of hydrogen the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11} m, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that v << c for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume that each revolution is essentially circular.)

The attractive force if coulombic:

$F = m a = \frac{1}{4 \pi \epsilon_0} \frac{q^2}{r^2} = m a = m \frac{v^2}{r}$. The last equality is due to a circular orbit

$v = \sqrt{\frac{1}{4 \pi \epsilon_0} \frac{q^2}{m r}}$. At the we start at $r_0 = 5 \times 10^{-11}$ meter, so:

$$\frac{v_0}{c} = \frac{1}{3 \times 10^8} \sqrt{\frac{(1.6 \times 10^{-19})^2}{4 \pi (8.85 \times 10^{-12})(9.11 \times 10^{-31})(5 \times 10^{-11})}}$$
$$= 0.0075$$

So it's starting as a pretty small fraction of c

To even get to 1% c, the radius would need to go to: $r \to 2.80754 \times 10^{-13}$

So we get to 1%c when r is ~ 3/500ths of its original value. So much of its trajectory is less then the speed of light.

To find the lifetime, we consider how much time it would take to radiate all the initial energy in the system.

We find the energy:

$E = U_{\text{kin}} + U_{\text{pot}} = \frac{1}{2}\, m\, v^2 - \frac{1}{4\,\pi\,\epsilon_0}\;\frac{q^2}{r}$
$= \frac{1}{2}\left(\frac{1}{4\,\pi\,\epsilon_0}\;\frac{q^2}{r}\right) - \frac{1}{4\,\pi\,\epsilon_0}\;\frac{q^2}{r}$
$= -\frac{1}{8\,\pi\,\epsilon_0}\;\frac{q^2}{r}.$

The energy loss per unit time must equal the power radiated:

$-\frac{dE}{dt} = \frac{-1}{8\,\pi\,\epsilon_0}\;\frac{q^2}{r^2}\;\frac{dr}{dt} = P = \frac{\mu_0\, q^2}{6\,\pi\, c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0\, q^2}{6\,\pi\, c}\left(\frac{1}{4\,\pi\epsilon_0}\;\frac{q^2}{m\, r^2}\right)^2 \Longrightarrow$
$\frac{-12\,\pi^2\; c\,\epsilon_0{}^2\; m^2\; r^2}{\mu_0\;\epsilon_0\; q^4}\, dr = dt \Longrightarrow$

We have a lifetime of:

$\frac{-12\,\pi^2\; c\,\epsilon_0{}^2\; m^2}{\mu_0\;\epsilon_0\; q^4}\int_{r_0}^{0} r^2\, dr = \int_0^{t_f} dt \Longrightarrow$
$\frac{4\,\pi^2\; c^3\;\epsilon_0{}^2\; m^2}{q^4}\; r_0{}^3 = \int_0^{t_f} dt = t_f$
$= 1.31543 \times 10^{-11}$ seconds

Scratch