*Assignment 7*
*Ph 110b*
*TA: John Joseph M. Carrasco*
*jjmc@physics.ucla.edu*

## Problem 1, Griffiths 12.10

A sailboat is manufactured so that the mas leans at an angle $\overline{\theta}$ with respect to the deck. An observer standing on a dock sees the boat go by at a speed $v$. What angle does this observer say the mast makes?

Let's give the mast unit length, such that the height in the boat's reference frame is $\sin(\overline{\theta})$, and it's horizontal projection is $\cos(\overline{\theta})$. When moving horizontally the height isn't affected, but the horizontal projection is Lorentz contracted to $\frac{1}{\gamma}\cos(\overline{\theta})$.

So: $\tan(\theta) = \frac{\sin(\overline{\theta})}{\frac{1}{\gamma}\cos(\overline{\theta})} = \gamma\tan(\overline{\theta})$.

So the observer on the dock sees: $\theta = \arctan(\gamma\tan(\overline{\theta}))$

## Problem 2, Griffiths 12.20

### ■ a.

Event $A$ happens at point {15, 5, 3, 0} and event B at {5,10,8,0} (using {ct,x,y,z} as Spacetime coordinates).

*i)* What is the invariant interval between A and B?

Griffith's uses the convention $a_\mu b^\mu = -a^0 b^0 + \vec{a}\cdot\vec{b}$, so we'll follow it. (I'm used to the spatial expression being assigned the negative sign, but OK).

$I = (A^\mu - B^\mu)(A_\mu - B_\mu) = -10^2 + 5^2 + 5^2 = 50 - 100 = -50.$

*ii)* Is there an inertial system in which they occur simultaneously? If so, find its velocity relative to $S$.

No. $I$ is timelike.

*iii)* Is there an inertial system in which they occur at the same point? If so, find its velocity relative to $S$.

Yes. Have $\tilde{S}$ travel in the direction from B towards A covering the distance in time $\frac{A^0 - B^0}{c} = \frac{10}{c} \Longrightarrow$
$\boldsymbol{v} = \frac{-5\hat{x} - 5\hat{y}}{10/c} = -\frac{c}{2}\hat{x} - \frac{c}{2}\hat{y}.$

### ■ b.

Event $A$ happens at point {1, 2, 0, 0} and event B at {3,5,0,0} (using {ct,x,y,z} as Spacetime coordinates).

*i)* What is the invariant interval between A and B?

Same type of thing as above:

$I = (A^\mu - B^\mu)(A_\mu - B_\mu) = -4 + 9 = 5$

*ii)* Is there an inertial system in which they occur simultaneously? If so, find its velocity relative to $S$.

Yes (I is spacelike). By lorentz transformation:

$\overline{(A^0 - B^0)} = \gamma((A^0 - B^0) - \beta(A^1 - B^1)) = 0 \Longleftrightarrow (A^0 - B^0) = \beta(A^1 - B^1) \Longleftrightarrow$

$\frac{(A^0 - B^0)}{(A^1 - B^1)} = \beta = \frac{v_x}{c} \Longrightarrow v_x = c\,\frac{(A^0 - B^0)}{(A^1 - B^1)} = \frac{2}{3}\,c$ .

*iii)* Is there an inertial system in which they occur at the same point? If so, find its velocity relative to $S$.

Nope, as we've a spacelike interval this time.

## Problem 3.

Using the Lorentz transformation of the four-velocity 4-vector, derive Einstein's rule for the addition of velocities directed along the transformation (Eq. 12.3).

Let's take a second to consider addition of velocities classically

Let's say within a lab there's a train is going north at some velocity $v$, and I'm goin' south at some velocity $u$. Classically, in my reference frame, I see the train moving north at velocity $v + u$. We can achieve this result by applying a classical gallilean transformation on the velocity $v$ to the reference frame moving at velocity $-u$ relative to the lab.

Now, from eqn 12.3 we know that the train should really (as in using Special Relativity) look like its moving north at velocity: $\frac{v+u}{1+\frac{vu}{c^2}}$ . We're going to get to this result by applying a lorentz transform on the velocity $v$ to get it into my reference frame (moving at velocity $-u$ relative to lab). Now to do a lorentz transform we'll need a 4 vector, and the natural 4-vector associated with velocity is the *proper 4-velocity*, usually referred to as simply as the 4-velocity.

So $\eta_v = \gamma_v\,\{c,\ v, 0, 0\}$ is the proper 4-velocity associated with the velocity $v$ given in section 12.2.1, where $\gamma_v \equiv \frac{1}{\sqrt{1-v^2/c^2}}$.

The lorentz transform to take us to a frame moving $-u$ is parameterized by: $\gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$, and $\beta_u = -\frac{u}{c}$.

Using $\Lambda = \begin{pmatrix} \gamma_u & -\gamma_u\,\beta_u & 0 & 0 \\ -\gamma_u\,\beta_u & \gamma_u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we find:

$\tilde{\eta_v} = \Lambda\,\eta_v = \gamma_u\,\gamma_v\,\{c - \beta_u\,v,\ -\beta_u\,c + v, 0, 0\}$

Now all we need to do is get our 3 velocity back out of our 4-velocity, which we can get by inverting eqn 12.40 ($\vec{\eta} = \frac{1}{\sqrt{1-v^2/c^2}}\,\vec{v}$): $\vec{v} = \frac{1}{\sqrt{1+\eta^2/c^2}}\,\vec{\eta}$ .

So $\tilde{v}_x = \frac{1}{\sqrt{1+\tilde{\eta}^2/c^2}}\,\tilde{\eta}_x = \frac{\gamma_u\,\gamma_v(-\beta_u\,c+v)}{\sqrt{1+(\gamma_u\,\gamma_v)^2\,\frac{(-\beta_u\,c+v)^2}{c^2}}}$

Now all we need to do is substitute in and giggle the algebra:

$$\tilde{v}_x = \frac{\gamma_u\,\gamma_v(-\beta_u\,c+v)}{\sqrt{1+(\gamma_u\,\gamma_v)^2\,\frac{(-\beta_u\,c+v)^2}{c^2}}} = \frac{-\beta_u\,c+v}{\sqrt{\left(\frac{1}{\gamma_u\,\gamma_v}\right)^2+\frac{(-\beta_u\,c+v)^2}{c^2}}}$$

$$= \frac{u+v}{\sqrt{(1-u^2/c^2)\,(1-v^2/c^2)+\frac{(u+v)^2}{c^2}}} = \frac{u+v}{\sqrt{\frac{u^2\,v^2}{c^4}+\frac{2\,u\,v}{c^2}+1}}$$

$$= \frac{u+v}{\sqrt{\frac{(c^2+u\,v)^2}{c^4}}} = \frac{u+v}{1+\frac{u\,v}{c^2}}$$

and we're done.

# Problem 4.

The high-energy linear accelerator at Stanford has a length of $L$ = 3 km, and uses electric fields in electromagnetic cavities to accelerate electrons in one direction (take it to be $x$) to 50 GeV. This final energy exceeds the rest energy of the electron (.511 MeV) by five orders of magnitude. The avg electric force performing work on the electrrons in the accelerator is $-e\,E_{\text{avg}} = 16.7\ \text{MeV}/m$, and thus the energy of the electron can be written as $E = -e\,E_{\text{avg}}\,x + m_e\,c^2$, or $\gamma(x) = 1 + \gamma'\,x$ where

$\gamma' = -\left(\frac{e\,E_{\text{avg}}}{m_e\,c^2}\right) = 32.7\,m^{-1}$. Note that with knowledge of

$\gamma(x) = \frac{1}{\sqrt{1-\frac{u(x)^2}{c^2}}}$ one may easily deduce what the velocity is as a function of $x$.

a) Show by integrating the velocity over $x$ that the time for the electrons to go from rest to final energy is:

$$t_f = \frac{L}{c}\left[\sqrt{\frac{\gamma_f+1}{\gamma_f-1}}\,\right] \simeq \frac{L}{c}$$

This we can get pretty simply:

$$\gamma(x) = \frac{1}{\sqrt{1-\frac{u(x)^2}{c^2}}} \implies$$

$$u(x) = c\sqrt{1-\frac{1}{(\gamma(x))^2}} = \frac{dx}{dt}$$

So:

$$t_f = \int_0^{t_f} dt = \int_1^{\gamma_f} \frac{1}{c\sqrt{1-\frac{1}{\gamma^2}}} \frac{d\gamma}{\gamma'} = \frac{1}{c\gamma'}\left[\sqrt{\gamma^2-1}\ \right]_0^{\gamma_f}$$

$$= \frac{1}{c\gamma'}\sqrt{\gamma_f{}^2-1} = \frac{1}{c\gamma'}\sqrt{(1+L\gamma')^2-1}$$

$$= \frac{1}{c\gamma'}\sqrt{L^2\gamma'^2+2L\gamma'} = \frac{1}{c\gamma'}\sqrt{\frac{L^3\gamma'^2+2L^2\gamma'}{L}}$$

$$= \frac{L}{c\gamma'}\sqrt{\frac{L\gamma'^2+2\gamma'}{L}} = \frac{L}{c}\sqrt{\frac{L\gamma'+2}{L\gamma'}}$$

$$= \frac{L}{c}\sqrt{\frac{\gamma_f+1}{\gamma_f-1}}$$

## ■ b

Show that the proper time for the elctrons to go from rest to final energy is: $\frac{L\cosh^{-1}(\gamma_f)}{c(\gamma_f-1)}$

$$\frac{dx}{dt} = \frac{dx}{\gamma\, d\tau} = u \Longrightarrow$$

$$\frac{1}{\gamma u}\, dx = d\tau \quad \Longrightarrow$$

$$\frac{1}{c\sqrt{\gamma^2-1}}\, dx = d\tau \quad \Longrightarrow$$

$$\frac{1}{c\sqrt{\gamma^2-1}} \frac{d\gamma}{\gamma'} = d\tau \quad \Longrightarrow$$

$$\int_{\gamma(0)}^{\gamma(L)=\gamma_f} \frac{1}{c\sqrt{\gamma^2-1}} \frac{d\gamma}{\gamma'} = \int_0^{\tau_f} d\tau = \tau_f \quad \Longrightarrow$$

$$\frac{\log\left(\gamma_f+\sqrt{\gamma_f^2-1}\right)}{c\gamma'} = \tau_f \Longrightarrow$$

$$\frac{\log\left(\gamma_f+\sqrt{\gamma_f^2-1}\right)}{c(\gamma_f-1)/L} = \tau_f \Longrightarrow$$

$$\frac{L\log\left(\gamma_f+\sqrt{\gamma_f^2-1}\right)}{c(\gamma_f-1)} = \tau_f$$

As $\cosh^{-1}(z) \equiv \ln\left(z+\sqrt{z^2-1}\right)$, we're done.

## ■ c

What is the length of the accelerator as seen by the electrons at final energy?

We can simply use our expression for lorentz contraction (12.9)

$$\Delta x = \sqrt{1-\frac{u(L)^2}{c^2}}\ (\Delta\,\overline{x} \equiv L) = \frac{1}{\gamma(L)}\, L = \frac{L}{(1+\gamma' L)} = \frac{3\times10^3}{(1+32.7\times3\times10^3)}$$

$$= 0.0305807\ m$$

## Problem 5.

Show that Griffiths, Eqs. 12.108, is consistent with the general relations governing transformation of the fields:

$\vec{E}_{\text{perp}}' = \gamma(\vec{E}_{\text{perp}} + \vec{v}\times\vec{B}_{\text{perp}})$

$\vec{B}_{\text{perp}}' = \gamma(\vec{B}_{\text{perp}} - \vec{v}\times\vec{E}_{\text{perp}})$

And parallel components unchanged.

In eqns 12.108, the velocity has only an $x$ component, and we see immediately that $E_x' = E_x$ and $B_x' = B_x$. Both $y$ and $z$ components transform according to the rules above as follows:

$E_y' \equiv \vec{E_y}' \cdot \hat{y} = \gamma(E_y - v B_z) = \gamma(\vec{E}\cdot\hat{y} + (v\,\hat{x}\times(B_z\,\hat{z} + B_y\,\hat{y} + B_x\,\hat{x}))\cdot\hat{y})$
$= \gamma(\vec{E}_{\text{perp}} + \vec{v}\times\vec{B}_{\text{perp}})\cdot\hat{y}$

$B_y' = \gamma(B_y + \frac{v}{c^2} E_z)$
$= \gamma(\vec{B}\cdot\hat{y} - (\frac{v}{c^2}\,\hat{x}\times(E_z\,\hat{z} + E_y\,\hat{y} + E_x\,\hat{x}))\cdot\hat{y})$
$= \gamma(\vec{B}_{\text{perp}} - \vec{v}\times\vec{E}_{\text{perp}})\cdot\hat{z}$

$E_z$ and $B_z$ follow by cyclic permutation of indices.

## Problem 6.

Consider a stationary uniform cylinder of charge that has free charge density $\rho = \rho_0(1-(s/a)^2)$ up to a radius $a$ ( $\rho_0$ is a constant ), and then vanishes outside of this radius.

### ■ a)

From Gauss's law, find the radial electric field associated with this charge distribution.

So Gauss's law gives us: $2\pi s\, E_s = 2\pi \frac{1}{\epsilon_0}\int_0^s \rho_0(1-(s/a)^2)\, s\, ds = 2\pi\frac{\rho_0}{\epsilon_0}\left(\frac{s^2}{2} - \frac{s^4}{4a^2}\right)$

allowing us to find $E_s = \frac{\rho_0}{2\epsilon_0}\left(s - \frac{s^3}{2a^2}\right)$ for $s < a$, and outside we have: $E_s = \frac{\rho_0}{4\epsilon_0}\frac{a^2}{s}$.

### ■ b)

Now assume that this distribution is accelerated to be a "beam" — it is in motion along its symmetry axis ($z$) with speed $v$ with respect to the lab frame.

From Lorentz transformation-derived rules for determining the fields, find the electric and magnetic fields in the lab frame associated with the moving charge distribution for the region $s < a$.

Well, let's note that $B_{x,y,z} = 0 = E_z$.

$E_z' = B_z' = 0$.

$E_s' = \gamma\left(E_s + \vec{v} \times \left(B_s\, \hat{s} + B_\phi\, \hat{\phi}\right) \cdot \hat{s}\right) = \gamma\, E_s$

$B_\phi' = \gamma\left(B_\phi - \frac{\vec{v}}{c^2} \times \left(E_s\, \hat{s} + E_\phi\, \hat{\phi}\right) \cdot \hat{\phi}\right) = \gamma\, (\frac{v}{c^2}\, E_s (\hat{z} \times \hat{s})) = \frac{\gamma\, v}{c^2}\, E_s\, \hat{\phi}$

$E_\phi' = B_s' = 0$.

So for $s < a$:

$\vec{E}' = \gamma\left(\frac{\rho_0}{2\, \epsilon_0}\, \left(s - \frac{s^3}{2\, a^2}\right)\right) \hat{s}$ and

$\vec{B}' = \gamma\left(\frac{\rho_0}{2\, \epsilon_0}\, \left(s - \frac{s^3}{2\, a^2}\right)\right)\, \frac{\gamma\, v}{c^2}\, \hat{\phi}$

## ■ c)

From Lorentz transformation of the charge-current 4-vector $\{\rho\, c, \vec{J}\}$, find the density $\rho'$ and current density $J_z'$ associated with the moving charge distribution.

Our 4-current is simply: $J \equiv \{c\, \rho_0 (1 - (s/a)^2), 0, 0, 0\}$, and our lorentz transformation parameterized by $\gamma = (1 - v^2/c^2)^{-1/2}$, and $\beta = v/c$ is given by:

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\gamma\, \beta \\ 0 & 0 & -\gamma\, \beta & \gamma \end{pmatrix}$$

so $J' = \gamma\, \{\, c\, p, 0, 0, -\beta\, c\, p\}$

## ■ d)

It's the same integral as in $a$) except now we've a $\gamma$ in front of our density$\Longrightarrow$

$E_s' = \gamma\, \frac{\rho_0}{2\, \epsilon_0}\, \left(s - \frac{s^3}{2\, a^2}\right) = \gamma\, E_s$ for $s < a$, and outside we have: $E_s' = \gamma\, \frac{\rho_0}{4\, \epsilon_0}\, \frac{a^2}{s} = \gamma\, E_s$.

Applying ampere's law to a circle of radius $s$ about the z-axis gives us:

$B_\phi' \times 2\, \pi\, s = -\mu_0\, \gamma\, v \int_0^s ds' (2\, \pi)\, \rho_0 (1 - (s/a)^2)$

$= -\mu_0\, \gamma\, v \int_0^s ds' (2\, \pi)\, \rho_0 (1 - (s/a)^2)$

$= -\mu_0\, \gamma\, v\, 2\, \pi\, \rho_0\, \left(\frac{s^2}{2} - \frac{s^4}{4\, a^2}\right)$

So for $s < a$: $B_\phi' = -\mu_0\, \gamma\, v\, \rho_0 \left(\frac{s}{2} - \frac{s^3}{4\, a^2}\right)$

$= -\frac{\gamma\, v}{c^2}\, E_s$

as found above.

## ■ e)

What is the net radial force on a particle of charge $q$ inside the beam moving with velocity u=v parallel to the z-axis?

If it's in the same reference frame as the "stationary" uniform cylinder of charge then the force would simply be:
$F_s = q\,E_s = q\,\frac{\rho_0}{2\,\epsilon_0}\left(s - \frac{s^3}{2\,a^2}\right)$

## Problem 7

Calculate the scalar potential $V$ for the charge distribution for the electric field calculated in Prob. 6, for the region $s < a$. By Lorentz boosting the potential 4-vector: $\{V/c, \vec{A}\}$ into the one with velocity $v$ along $z$ find $V'$ and A'. What are the electric and magnetic fields due to these potentials in the moving frame?

We find the potential by integrating:

$V = -\int_O^s E(s')\,ds' = \frac{s^2\,(s^2-4\,a^2)\,\rho_0}{16\,a^2\,\epsilon_0}$

The vector potential is zero in the rest frame, so we have the four potential: $\left\{\frac{1}{c}\,\frac{s^2\,(s^2-4\,a^2)\,\rho_0}{16\,a^2\,\epsilon_0},\,0,\,0,\,0\right\}$.

We take the same lorentz transformation used in problem 6 to boost the current 4-vector, and find:

$$A' = \Lambda\,A$$
$$= \gamma\left\{\frac{1}{c}\,\frac{s^2\,(s^2-4\,a^2)\,\rho_0}{16\,a^2\,\epsilon_0},\;0,\;0,\;-\beta\,\frac{1}{c}\,\frac{s^2\,(s^2-4\,a^2)\,\rho_0}{16\,a^2\,\epsilon_0}\right\}$$
$$= \gamma\left\{\frac{1}{c}\,\frac{s^2\,(s^2-4\,a^2)\,\rho_0}{16\,a^2\,\epsilon_0},\;0,\;0,\;-\frac{v}{c^2}\,\frac{s^2\,(s^2-4\,a^2)\,\rho_0}{16\,a^2\,\epsilon_0}\right\}$$

We calculate $F^{\mu\nu}$ using 12.132: $F^{\mu\nu} = \partial_\mu\,A^\nu - \partial_\nu\,A^\mu$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \frac{x\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c\,\epsilon_0} & \frac{y\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c\,\epsilon_0} & 0 \\ -\frac{x\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c\,\epsilon_0} & 0 & 0 & \frac{v\,x\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c^2\,\epsilon_0} \\ -\frac{y\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c\,\epsilon_0} & 0 & 0 & \frac{v\,y\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c^2\,\epsilon_0} \\ 0 & -\frac{v\,x\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c^2\,\epsilon_0} & -\frac{v\,y\,(-2\,a^2+x^2+y^2)\,\gamma\,\rho_0}{4\,a^2\,c^2\,\epsilon_0} & 0 \end{pmatrix}$$

which gives us our fields.

## Problem 8, Griffiths 12.43

### a)

Check that Gauss's law $\oint E \cdot da = \frac{1}{\epsilon_0} Q_{\text{enc}}$ is obeyed by the field of a point charge in uniform motion, by integrating over a sphere of radius $R$ centered on that charge.

$$E = \frac{1}{4\pi\epsilon_0} \frac{q(1-v^2/c^2)}{(1-v^2/c^2 \sin^2(\theta))^{3/2}} \frac{\hat{R}}{R^2} \text{(Eq. 12.92)} \Longrightarrow$$

$$\int E \cdot da = \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} \int \frac{d\theta\, d\phi \sin(\theta) R^2}{R^2 (1-v^2/c^2 \sin^2\theta)^{3/2}}$$

$$= \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} 2\pi \int_0^\pi \frac{d\theta \sin(\theta)}{(1-v^2/c^2 \sin^2\theta)^{3/2}}$$

$$= \frac{q(1-v^2/c^2)}{2\epsilon_0} \int_{-1}^{+1} \frac{du}{\left(1-v^2/c^2+\frac{v^2}{c^2} u^2\right)^{3/2}}$$

$$= \frac{q(1-v^2/c^2)}{2\epsilon_0} \left(\frac{c}{v}\right)^3 \int_{-1}^{+1} \frac{du}{\left(\frac{c^2}{v^2}-1+u^2\right)^{3/2}}$$

$$= q \frac{(1-v^2/c^2)}{2\epsilon_0} \left(\frac{c}{v}\right)^3 \left(\frac{v}{c}\right)^3 \frac{2}{(1-v^2/c^2)} = q$$

## b)

Find the Pointing vector for a point charge in uniform motion. (Say the charge is going in the $z$ direction at speed $v$, and calculate $\boldsymbol{S}$ at the instant $q$ passes the origin.

$$\boldsymbol{S} = \frac{1}{\mu_0} (\boldsymbol{E} \times \boldsymbol{B}) = \frac{1}{\mu_0} \left( \frac{q(1-v^2/c^2)}{(1-v^2/c^2 \sin^2(\theta))^{3/2}} \frac{\hat{R}}{R^2} \times \frac{\mu_0}{4\pi} \frac{q\, v(1-v^2/c^2) \sin(\theta)}{(1-v^2/c^2 \sin^2(\theta))^{3/2}} \frac{\hat{\phi}}{R^2} \right)$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{1}{4\pi R^4} \frac{v\, q^2 (1-v^2/c^2)^2 \sin(\theta)}{(1-v^2/c^2 \sin^2(\theta))^3} \hat{\theta}$$