TA Section 3

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Rel. E field from Mag Field

Ok, consider two parallel plates separated by some Δz both with charge σ, with currents (top/bottom) $j_{\binom{t}{b}} = \pm \sigma\, u_x$.

What is the magnetic field in our original reference frame? The magnetic field is given by ampere's law. Consider the top plate contribution:
$\oint \vec{B} \cdot d\vec{l} = 2\,B\,l = \mu_0\,K\,l \implies B_y = \frac{\mu_0}{2}\,K = \frac{\mu_0}{2}\,\sigma\,u$. Both plates constructively superimpose in the middle region giving us: $B_y = \mu_o\,\sigma\,u_x$.

What is Electric field in this reference frame? 0 from symmetry.

So what is $\vec{E}'$ in reference frame moving with some v in the x direction?

Well, realize that in our original reference frame $\sigma_{\text{top}} = \frac{1}{\sqrt{1-u_t^2/c^2}}\,\sigma_0$, and $\sigma_{\text{bottom}} = \frac{1}{\sqrt{1-u_b^2/c^2}}\,\sigma_0$.

In new reference frame: $\sigma_{\text{top}}' = \frac{1}{\sqrt{1-(u_t'/c)^2}}\,\sigma_0$, and $\sigma_{\text{bottom}}' = \frac{1}{\sqrt{1-(u_b'/c)^2}}\,\sigma_0$. $u_t' = \frac{u_t - v}{1-\frac{u_t\,v}{c^2}} = \frac{u-v}{1-\frac{u\,v}{c^2}}$.

$u_b' = \frac{u_t - v}{1-\frac{u_t\,v}{c^2}} = -\frac{u+v}{1+\frac{u\,v}{c^2}}$.

So $\sigma_{\text{top}}' = \frac{\sigma_0}{\sqrt{1-\left(\frac{u-v}{1-\frac{u\,v}{c^2}}\right)^2 / c^2}}$, similarly $\sigma_{\text{bottom}}' = \frac{\sigma_0}{\sqrt{1-\left(\frac{u+v}{1+\frac{u\,v}{c^2}}\right)^2 / c^2}}$.

$$\text{So: } E_z' = \frac{\sigma_b'}{2\,\epsilon_0} - \frac{\sigma_t'}{2\,\epsilon_0} = \frac{\sigma_0}{2\,\epsilon_0}\left(\frac{1}{\sqrt{1-\left(\frac{u+v}{1+\frac{u\,v}{c^2}}\right)^2 / c^2}} - \frac{1}{\sqrt{1-\left(\frac{u-v}{1-\frac{u\,v}{c^2}}\right)^2 / c^2}}\right)$$

$$= \frac{u\,v\,\sigma_0/\epsilon_0}{\sqrt{(c^2-u^2)(c^2-v^2)}\;\epsilon_0} = \frac{1}{\epsilon_0}\,\gamma_v\,\beta_v\;\frac{u}{c}\,\gamma_u\,\sigma_0$$

$$= \gamma_v\,\beta_v\,\frac{1}{\epsilon_0\,c}\,\sigma\,u = \gamma_v\,\beta_v\,\frac{1}{\epsilon_0}\,\frac{1}{\mu_0}\,\frac{1}{c}\,B_y$$

$$= \gamma_v\,\beta_v\,c^2\,\frac{1}{c}\,B_y = \gamma_v\,v_x\,B_y$$

We see this satisfies the general transformation rules given in table 12.108:
For a reference frame moving in direction x_i with velocity v relative to our original:

$\overline{E_i} = E_i$, $\overline{E}_{i+1} = \gamma(E_{i+1} - v\,B_{i+2})$, $\overline{E}_{i+2} = \gamma(E_{i+2} + v\,B_{i+1})$

$\overline{B_i} = B_i$, $\overline{B}_{i+1} = \gamma\left(B_{i+1} + \frac{v}{c^2}\,E_{i+2}\right)$, $\overline{B}_{i+2} = \gamma\left(B_{i+2} - \frac{v}{c^2}\,E_{i+1}\right)$

I wrote this slightly more abstractly then the table in the book as it satisfies cyclic permutations of the index. i.e. for v in the x direction: $i = 1 \Longrightarrow$ $E_i = E_x$, $E_{i+1} = E_y$, $E_{i+2} = E_z$. For v in the y direction: i=2 $\Longrightarrow$ $E_i = E_y$, $E_{i+1} = E_z$, $E_{i+2} = E_x$ (Note I wrapped i+2=4 back around to 1).

- **EASY WAY (if you remember how B transforms)**

$E_i\,B^i$ is conserved

$\frac{E^2}{c^2} - B^2$ is conserved

Ok, if you remember that $\vec{B}' = \gamma\,B_y\,\hat{y}$, given no electric field in original reference frame:

We know that $E_i\,B^i = 0$ in original reference frame, so E' has to point in either z direction or x direction, but symmetry rules out x, so we know only z direction.

We then have: $\frac{E_z'^2}{c^2} - \gamma^2\,{B_y}^2 = -{B_y}^2 \Longrightarrow E_z' = c\,B_y\left(\gamma^2 - 1\right)^{1/2} = c\,B_y\,\gamma\,\beta = v\,\gamma\,B_y$

Relative forces (Griffiths 12.44)

Charge q_A is at rest at the origin in system S; charge q_B flies by at speed v on a trajectory parallel to the x axis, but at y=d. What is the electromagnetic force on q_B as it crosses the y axis?

Soln: Fields of particle A at B: $E = \frac{1}{4\pi\epsilon_0}\,\frac{q_A}{d^2}\,\hat{y}$; B=0 $\Longrightarrow F = \frac{1}{4\pi\epsilon_0}\,\frac{q_A\,q_B}{d^2}\,\hat{y}$

Now look at the problem from system $\overline{S}$ which moves along the x axis w/ speed v. What is the force on q_B when q_A passes the $\overline{y}$ axis. Do it two ways: 1) use the answer we just got and transform the force. 2) compute the fields in $\overline{S}$ and use the Lorentz force law.

1) From 12.68: $\overline{F} = \frac{\gamma}{4\pi\epsilon_0}\,\frac{q_A\,q_B}{d^2}$. (worth looking at the derivation on pg 518, but no time for it now. By the way, before the midterm, you should be able to derive 12.73 by hand for every component as it was a homework problem!)

2) From Eqn. 12.92, w/ θ=π/2.

$\overline{E} = \frac{1}{4\pi\epsilon_0}\,q_A\,\frac{(1-v^2/c^2)}{(1-v^2/c^2)^{3/2}}\,\frac{1}{d^2}\,\hat{y} = \frac{\gamma}{4\pi\,\epsilon_0}\,\frac{q_A}{d^2}\,\hat{y}$. Note that $\overline{B} \neq 0$, but since $v_B = 0$ in $\overline{S}$ there is no magnetic force $(0\times\overline{B} = 0)$ so we achieve the same result.

Kinematics. pair-annihilation (Griffiths 12.35)

In a pair annihilation experiment, an electron (m) with momentum (p_e) hits a positron (same mass, but opposite charge) at rest. They annihilate, producing two photons. (Why couldn't they produce just one photon?) If one of the photons emerges at 60° to the incident electron direction, what is its energy?

Answer:

One photon is impossible. Consider the center of momentum frame, where the total momentum is zero.
We'd be left with one photon at rest in this frame, but photons have to travel at the speed of light, so we need at least two photons for a center of momentum frame to exist after the collision.

Ok. so before we have: $m \longrightarrow$ m.

After we have two photons: one with energy E_A going up at 60°, and another photon with energy E_B going down at some angle θ.

What two things are conserved? Energy and momentum!

Well, generically, given a four momentum $p^\mu = \left(E/c, \vec{p}\right)^\mu$, and the fact that $p^\mu p_\mu = -m^2 c^2$, we have: $E^2 = m^2 c^4 + p^2 c^2 \Longrightarrow$ The energy of a particle moving with momentum p and of rest mass m is: $\sqrt{p^2 c^2 + m^2 c^4}$. So conservation of energy tells us:

$$\sqrt{p_e{}^2 c^2 + m^2 c^4} + m c^2 = E_A + E_B$$

Conservation of momentum gives us that:
sideways:

$$p_e = \frac{E_A}{c} \cos(60\,°) + \frac{E_B}{c} \cos(\theta) \Longrightarrow E_B \cos(\theta) = p_e c - \frac{1}{2} E_A$$

up/down:

$$0 = \frac{E_A}{c} \sin(60\,°) - \frac{E_B}{c} \sin(\theta) \Longrightarrow E_B \sin(\theta) = \frac{\sqrt{3}}{2} E_A$$

Squaring and adding we find:

$$E_B{}^2 = (p_e c)^2 + \frac{1}{4} E_A{}^2 - E_A p_e c + \frac{3}{4} E_A{}^2$$
$$= (p_e c)^2 + E_A{}^2 - E_A p_e c$$

Which is equal to from conservation of energy:

$$= \left(\sqrt{p_e c^2 + m^2 c^4} + m c^2 - E_A\right)^2$$

Well, now we've just got a quadratic equation (if a little messy) in terms of E_A and the problem can be solved straightforwardly.

HW Hints.

4) Prove that $\eta_{\mu\nu} = \eta^{\mu\nu}$ is invariant under Lorentz transformations.

Due to symmetry it is sufficient to consider a Lorentz boost along the x direction and explicitly show equality, i.e. $\eta^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. This is the recommended approach for this problem.

Slightly more sophisticated we can use the invariance of the inner product under Lorentz transformation to show this in a couple of lines. Consider four vectors x, and y, with non-vanishing inner product. Invariance of the inner product means:
$x_\mu y^\mu = x'_\lambda y'^\lambda \Longleftrightarrow \eta_{\mu\nu} x^\mu y^\nu = \eta_{\lambda\rho} x'^\lambda y'^\rho$, where x'^α and y'^α are components in the boosted reference frame. Expanding out x' and y' in terms of explicit Lorentz boosts of x and y demonstrates the invariance of the metric.

5) Duality.

You want to use the following notion of duality: $G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$ where $\epsilon^{\mu\nu\rho\lambda}$ is the completely antisymmetric tensor analogous to the levi-Civita tensor in 3-D. Notice in the above expression for $G^{\mu\nu}$ on the right hand side indices ρ and λ are repeated and thus summed over from 0 to 3, but from the following table:

$\epsilon^{\mu\nu\rho\sigma}$ (4 D)	ϵ^{ijk} (3 D)
0 when any two indices are the same	0 when any two indices are the same
1 for even permutations of {0, 1, 2, 3} : (0123, 2301, ...)	1 for even permutations of {1, 2, 3} : (123, 231, ...)
−1 for odd permutations of {0, 1, 2, 3} : (0213, 1302, ...)	−1 for odd permutations of {1, 2, 3} : (213, 132, ...)

we see that ρ and λ can only each take on one of two values for non-vanishing contributions.

e.g. $\epsilon^{12\rho\lambda} F_{\rho\lambda} = F_{30} - F_{03}$.

Also recall that lowering a tensor index with a zero component involves the introduction of a minus sign.
i.e. $F_{23} = F^{23}$, but $F_{03} = -F^{03}$.

- **scratch**